Infinite horizon MPC applied to batch processes. Part I

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Abstract: In the first stage of this work (part I), an infinite horizon model predictive controller (IHMPC) using the closed-loop paradigm, was adapted to be applied to batch processes. Batch processes fall into the category of finite-duration systems, where the control objective is to track a given output profile (reference trajectory) during a finite time period, and so the use of an infinite horizon controller is not intuitive *a priori*. However, based on previous IHMPC results, this work shows that prediction along a fictitious horizon beyond the actual final batch time allows good performance and, more important, the stability (in the sense of the finite-duration systems) can be guaranteed. Simulation results show the closed-loop performance of the proposed strategy when a linear system is controlled. In addition, the proposed methodology allows a direct extension to a new MPC with learning properties - i.e. a MPC strategy that "learns" from previous batches - which accounts for the second important property required to batch processes: the repetitiveness. The way the proposed strategy is extended to the MPC with learning properties is presented in a second part of this work (González et al., 2009).

1. INTRODUCTION

In many approaches attempting to develop a strategy to handle batch processes, an optimal control is performed over a finite horizon, that is, over a horizon of the same length of the batch (Lee and Lee, 1997, 1999, 2000; Cueli and Bordons, 2008). The use of this finite horizon is intuitive and no other alternative has been explored at this point, since a larger – or shorter – horizon would be adding in principle no significant information for predictions. However, the use of typical MPC for continuous processes has shown that not always exact future information is required to achieve stability and acceptable performances. An interesting example of this properties is the original infinite horizon MPC, IHMPC (Gonzalez et al., 2008a), that assumes that the closed loop controller acting beyond the control horizon is the zero-controller (i.e., the state feedback K=0), which is not true even for the unconstrained case. Using this strategy, and depending on the reference trajectory form (slope and length), it is possible to design an acceptable (and closed loop stable) controller taking into account only several time instants ahead.

Now, to adapt the IHMPC controller to batch requirements, a virtual output horizon beyond the final batch time is assumed. This allows an intuitive comparison between the successive optimization costs, and so, it is possible to establish asymptotic convergence conditions for the successive costs of a given batch (intra-run convergence). The IHMPC, in addition, is formulated under a closed-loop paradigm (Rossiter, 2003). The basic idea of a closed-loop paradigm is to choose a stabilizing control law and assume that this law (underlying input sequence) is present throughout the predictions.

The proposed IHMPC considers an underlying (preliminary) control sequence as a (deficient) reference candidate for the tracking control. Then, by solving on line a constrained optimization problem, the input sequence is corrected.

The paper is organized as follows. In Section 2 the basic definition and notation are presented. Then, in Section 3, the proposed MPC formulation is introduced and the intra-run convergence analysis is discussed. Finally, two succinct illustrative examples and the conclusion are presented in sections 4 and 5, respectively.

2. PRELIMINARIES

We consider that the batch has a length of T_f time instants, and the objective is to find an input sequence defined by

$$\boldsymbol{u} := \begin{bmatrix} u_0^T & \cdots & u_{T_f-1}^T \end{bmatrix}^T, \tag{1}$$

which derives in an output sequence

$$\boldsymbol{y} := \begin{bmatrix} y_1^T & \cdots & y_{T_f}^T \end{bmatrix}^T, \tag{2}$$

as close as possible to a output reference trajectory

$$\boldsymbol{y}^r := \begin{bmatrix} y_1^{r^T} & \cdots & y_{T_f}^{r^T} \end{bmatrix}^T, \tag{3}$$

On the other hand, it is assumed that there exists an input reference sequence (an input candidate) given by

$$\boldsymbol{u}^r := \begin{bmatrix} u_0^{r^T} & \cdots & u_{T_f-1}^{r^T} \end{bmatrix}^T, \tag{4}$$

where $u^r_{T_f-1}$ represents a stationary input value, satisfying $u^r_{T_f-1}=G^{-1}(y^r_{T_f}-d_{T_f})$, with $G=[C(I-A)^{-1}B]$. The output disturbance,

$$\boldsymbol{d} := \begin{bmatrix} d_1^T & \cdots & d_{T_f}^T \end{bmatrix}^T,$$

is assumed to be known, and its final stationary value is given by d_{T_f}. The disturbance is assumed to remain constant along the run.

2.1. Nominal model

The model used to construct the forecast of the receding horizon policy is given by:

$x_{k+1}=Ax_k+Bu_k$ (5)

$d_{k+1}=d_k$ (6)

$y_k=Cx_k+d_k$ (7)

where A, B and C are matrices of appropriate dimension, x are the states and d is a constant output disturbance (González et al., 2008b).

3. BASIC FORMULATION

For the proposed MPC formulation we assume that an appropriate input reference $\boldsymbol{u}^r$ is available (otherwise, it is possible to use a null constant value since the formulation works anyway), and the disturbance $d_{k|k}$ as well as the states $x_{k|k}$ are estimated. Under these assumptions it is possible to formulate the problem P1 as follows:

Problem P1)

$$\min_{\left\{\overline{u}_{k|k},\cdots,\overline{u}_{k+T_f-1|k}\right\}} V_k = \sum_{j=0}^{T_f-1} \ell\left(e_{k+j|k},\overline{u}_{k+j|k}\right) + F\left(x_{k+T_f|k}\right)$$

subject to:

$$e_{k+j|k} = Cx_{k+j|k} + d_{k+j} - y^r_{k+j}, \quad j=0,\cdots,T_f,$$

$$x_{k+j+1|k} = Ax_{k+j|k} + Bu_{k+j|k}, \quad j=0,\cdots,T_f,$$

$$u_{k+j|k} \in U, \quad j=0,1,\cdots,N-1, \tag{8}$$

$$u_{k+j|k} = u^r_{k+j} + \overline{u}_{k+j|k}, \quad j=0,1,\cdots,T_f-1, \tag{9}$$

$$\overline{u}_{k+j|k} = 0, \quad j=N,N+1,\cdots,T_f-1, \tag{10}$$

where

$$u^r_k = u^r_{T_f-1} = G^{-1}\left(y^r_{T_f} - d_{T_f}\right), \quad k=T_f-1,T_f,\cdots,2T_f-2,$$

$$y^r_k = y^r_{T_f}, \quad k=T_f,T_f+1,\cdots,2T_f-1,$$

and

$$\ell(e,u) := \|e\|_Q^2 + \|u\|_R^2, \qquad F(x) := \left\|x - x_{T_f}\right\|_P^2,$$

$U=\{u: u_{min} \leq u \leq u_{max}\}$. Matrices Q, R and P are such that Q and P>0 and R≥0. The stationary state $x^r_{T_f}$ is such that $x^r_{T_f} = Ax^r_{T_f} + Bu^r_{T_f-1}$.

Remark 0: The decision variables $\overline{u}^i_{k+j|k}$, are a correction to the input reference sequence u^{ir}_{k+j} (see Eq. (9)), attempting to improve the closed-loop predicted performance. Because of constraints (10), $\overline{u}^i_{k+j|k}$ is different from zero only in the first N steps and so, the optimization problem P1 has N decision variables. The input and output references, u^{ir}_{k+j} and y^r_{k+j}, as well as the disturbance d^i_{k+j} are optimization parameters. The stationary reference state $x^{ir}_{T_f}$ is defined as $x^{ir}_{T_f} = (I-A)^{-1}Bu^{ir}_{T_f-1}$, which implies $y^r_{T_f} = Cx^{ir}_{T_f} + d^i_{T_f}$.

Remark 1: In a given batch iteration, T_f optimization problems P1 must be solved. Each problem gives an optimal input sequence $u^{opt}_{k+j|k}$, for $j=0,...,T_f-1$, and following the receding horizon policy, only the first input of the sequence, $u^{opt}_{k|k}$, is applied to the system.

Remark 2: The output or prediction horizon is given in principle by a T_f length, and so the predictions go up to $k+T_f$ (See Figure 1). However, as will be shown later, the terminal cost represents the tail of the predicted output from $k+T_f$ to infinity, which means that the complete output horizon is in fact infinite. In this context, since all the predicted variables beyond T_f are not based on actual variables (given that the process ends at T_f), the time interval from T_f to infinity is designed as *virtual output horizon for predictions*.

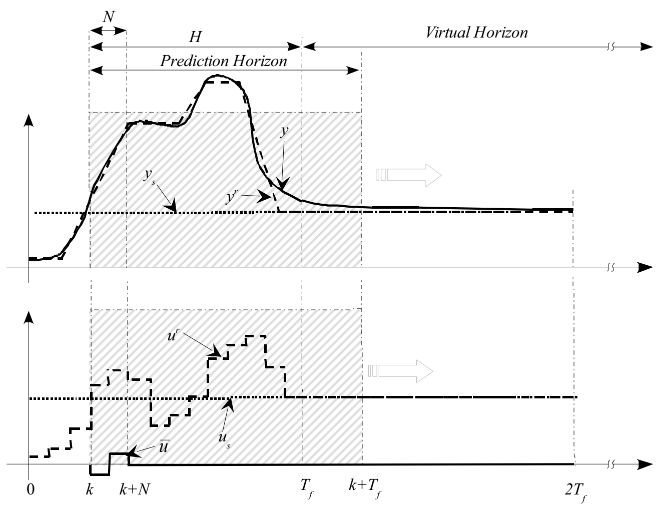

Fig. 1. MPC diagram.

Remark 3: Notice that with the disturbance assumptions, the cost is bounded, since for the model used to compute the predictions, the new steady state input and state will correspond to ($y^r_{T_f} - d^i_{T_f}$). Furthermore, since the model output is guided to ($y^r_{T_f} - d^i_{T_f}$), the system output will be guided to $y^r_{T_f}$ (assuming that the disturbances remain the same).

Remark 4: As usual in the MPC literature, the nomenclature $x_{k+j|k}$ stand for the predicted variable x_{k+j}, estimated at time k. In this way, predictions are given by

$x_{k+1|k}=Ax_{k|k}+Bu_{k|k}$

$x_{k+2|k}=A^2x_{k|k}+ABu_{k|k}+Bu_{k+1|k}$,

$\vdots$

Besides, under the condition that a perfect model is used and no unknown disturbance enter the system, it follows that $x_{k+1|k+1}=x_{k+1|k}=Ax_{k|k}+Bu_{k|k}$; but, in general, $x_{k+j|k+1}\neq x_{k+j|k}$ for $j\geq 2$.

3.1 Non-increasing properties of the closed loop cost, for a given batch (intra-run convergence)

The concept of stability for a finite-duration process is different from the traditional one since, except for some special cases such finite-time escape, boundless of the disturbance effect is trivially guaranteed. In Srinivasan and Bonvin (2007), the authors define a quantitative concept of stability by defining a *variability index* as the induced norm of the variation around a reference (state) trajectory, caused by a variation in the initial condition. Here, based on the idea of virtual horizon for predictions presented before, we will show two controller properties (Theorem 1). 1) The optimal IHMPC cost monotonically decreases with time k, and 2) if the control algorithm execution goes beyond T_f with $k\to\infty$, and the output reference remains constant at the final reference value ($y_k^r = y_{T_f}^r$ for $k\geq T_f$) then, the IHMPC cost goes to zero as $k\to\infty$, which implies that $y_k^i \to y_{T_f}^r$ as $k\to\infty$.

Theorem 1 (intra-run convergence)

Let assume that matrix P is such that

$$C^TQC-P+A^TPA\leq 0. \tag{11}$$

Then, assuming that the disturbance remains constant from one batch to the next, and the batch has an infinite hypothetic duration (i.e. if $k\to\infty$), the output error converges to zero.

Proof

To assure the stability of the closed loop we need to show that the optimal cost $V_k^{i\,opt}$ is a Lyapunov function. Let $\bar{\mathbf{u}}_{k-1}^{opt} := \left\{\bar{u}_{k-1|k-1}^{opt},\cdots,\bar{u}_{k+N-1|k-1}^{opt},0,\cdots,0\right\}$ and $\boldsymbol{x}_{k-1}^{opt} := \left\{x_{k-1|k-1}^{opt},\cdots,x_{k+T_f-1|k-1}^{opt}\right\}$ be the optimal input and state sequence that are the solution to problem P1 at time k-1. The cost corresponding to these variables are

$$V_{k-1}^{opt} = \sum_{j=0}^{N-1}\ell\left(e_{k+j-1|k-1}^{opt},\bar{u}_{k+j-1|k-1}^{opt}\right)+ \sum_{j=N}^{T_f-1}\ell\left(e_{k+j-1|k-1}^{opt},0\right)+F\left(x_{k+T_f-1|k-1}^{opt}\right) \tag{12}$$

Now, let $\bar{\mathbf{u}}_k^{feas} := \left\{\bar{u}_{k|k-1}^{opt},\cdots,\bar{u}_{k+N-2|k-1}^{opt},0,\cdots,0\right\}$ be a feasible solution to problem P1 at time k. Since no new input is injected to the system from time k-1 to time k, and no unknown disturbance is considered, the predicted state at time k, using the feasible input sequence, will be given by $\boldsymbol{x}_k^{feas} := \left\{x_{k|k-1}^{opt},\cdots,x_{k+T_f-1|k-1}^{opt},\left(Ax_{k+T_f-1|k-1}^{opt}+Bu_{T_f-1}^r\right)\right\}$ Then, the cost at time k corresponding to the feasible solution $\bar{\mathbf{u}}_k^{i\,feas}$ is as follows:

$$V_k^{feas} = \sum_{j=0}^{N-1}\ell\left(e_{k+j|k-1}^{opt},\bar{u}_{k+j|k-1}^{opt}\right)+ \sum_{j=N}^{T_f-1}\ell\left(e_{k+j|k-1}^{opt},0\right)+F\left(Ax_{k+T_f-1|k-1}^{opt}+Bu_{T_f-1}^r\right). \tag{13}$$

Now, subtracting (12) from (13) we have

$$\begin{aligned}V_k^{feas}-V_{k-1}^{opt} &= -\ell\left(e_{k-1|k-1}^{opt},\bar{u}_{k-1|k-1}^{opt}\right)-F\left(x_{k+T_f-1|k-1}^{opt}\right)\\ &\quad+\ell\left(e_{k+T_f-1|k-1}^{opt},0\right)+F\left(Ax_{k+T_f-1|k-1}^{opt}+Bu_{T_f-1}^r\right),\end{aligned} \tag{14}$$

and given that $x_{T_f}^r = Ax_{T_f}^r + Bu_{T_f-1}^r$, and $y_k^r = Cx_{T_f}^r + d_{T_f}$, for $k\geq T_f$, and taking into account the definition of the stage and terminal costs (ℓ and F), equation (14) can be written as

$$\begin{aligned}V_k^{feas}-V_{k-1}^{opt} &= -\ell\left(e_{k-1|k-1}^{opt},\bar{u}_{k-1|k-1}^{opt}\right)-\left\|x_{k+T_f-1|k-1}^{opt}-x_{T_f}^r\right\|_P^2\\ &\quad+\left\|Cx_{k+T_f-1|k-1}^{opt}-Cx_{T_f}^r\right\|_Q^2+\left\|Ax_{k+T_f-1|k-1}^{opt}+\overbrace{Bu_{T_f-1}^r-x_{T_f}^r}^{-Ax_{T_f}^r}\right\|_P^2\\ &= -\ell\left(e_{k-1|k-1}^{opt},\bar{u}_{k-1|k-1}^{opt}\right)+\left\|x_{k+T_f-1|k-1}^{opt}-x_{T_f}^r\right\|_{\left(C^TQC-P+A^TPA\right)}^2,\end{aligned} \tag{15}$$

Now, since by hypothesis matrix P holds (11) true, then the last term of the right hand side of (15) will be negative, and so, the difference between the feasible cost at time k and the optimal cost at time k-1 can be written as

$$V_k^{feas}-V_{k-1}^{opt} \leq -\ell\left(e_{k-1|k-1}^{opt},\bar{u}_{k-1|k-1}^{opt}\right). \tag{16}$$

This means that the optimal cost at time k, which is not greater than the feasible one at the same time, satisfy

$$V_k^{opt}-V_{k-1}^{opt}+\ell\left(e_{k-1|k-1}^{opt},\bar{u}_{k-1|k-1}^{opt}\right)\leq 0, \tag{17}$$

which shows that the cost is not increasing when the time k increases. In addition, if time k goes to infinity (which is not true in a batch process), the cost goes to zero. Despite the batch process ends at a finite time T_f, this result guarantees that the system will be stable in the sense of Lyapunov.□

Remark 5: Notice that $e_{k-1|k-1}^{opt}$ and $\bar{u}_{k-1|k-1}^{opt}$ represent the actual error and the implemented input, respectively. In order to make clear this fact we write

$$V_k^{opt}-V_{k-1}^{opt}+\ell\left(e_{k-1},\bar{u}_{k-1}\right)\leq 0. \tag{18}$$

Remark 6: We can propose different strategies to determine the terminal penalty form so that (11) holds true. It is useful at this point to assume that: 1) a terminal (or local) stabilizing controller acts on the (predicted) states from predictions time $j=T_f$ on, and 2) the terminal penalty is given by the cost of bringing the system from $j=T_f$ to infinite, using the local controller (that is why the controller is named IHMPC).

Notice that despite the batch process ends at a finite time T_f, the open loop prediction performed by the MPC algorithm can assume that prediction time j goes to infinity. Provided that the system is assumed to be open loop stable, one simple choice is a constant local controller given by: $u^i_{k+j|k} = u^{ir}_{T_f-1}$ for $j=T_f+1$, T_f+2,... In this case, the terminal penalty has the form

$$F\left(x_{k+T_f|k}\right) = \left\|x_{k+T_f|k} - x^r_{T_f}\right\|^2_P = \sum_{j=T_f}^{\infty} \left\|x_{k+j|k} - x^r_{T_f}\right\|^2_{C^TQC} . \quad (19)$$

So, since $x_{k+j+1|k} = Ax_{k+j|k} + Bu_{k+j|k} = Ax_{k+j|k} + Bu^r_{T_f-1}$, for $j=T_f$, T_f+1,..., and $x^r_{T_f} = Ax^r_{T_f} + Bu^r_{T_f-1}$, it follows that $x_{k+j+1|k} - x^r_{T_f} = A\left(x_{k+j|k} - x^r_{T_f}\right)$, which implies $P = \sum_{j=0}^{\infty} A^{jT} C^T QCA^j$,

and $A^T PA = \sum_{j=1}^{\infty} A^{jT} C^T QCA^j = P - C^T QC$. With this choice, inequality (16) becomes equality, and V_k^{opt} is a Lyapunov function of the closed loop.

Remark 7: Using the terminal penalty definition (19), the output horizon length of the proposed formulation becomes infinity. So, the *virtual output horizon for predictions*, that is, the horizon containing the predictions beyond the final time T_f, becomes infinity.

Remark 8: Now, let us define a horizon H being the distance between the current time k and the final time T_f (that is, $H:=T_f-k$). Therefore,

a) For $k=0,...,T_f-N-1$ (that is, $H>N$), and taking into account (19), the MPC optimization problem predictions can be divided into two periods of time. The first one is the time period $0\leq j\leq H$, where in general $u_{k+j|k} \neq u^r_{T_f-1}$ and $y^r_{k+j|k} \neq y^r_{T_f}$; and the second one is the time period $H\leq j<\infty$, where $u_{k+j|k} \neq u^r_{T_f-1}$ and $y^r_{k+j|k} \neq y^r_{T_f}$. As a result, the MPC cost at time k can be written as

$$\begin{aligned} V_k &= \overbrace{\sum_{j=0}^{N-1} \ell\left(e_{k+j|k}, \bar{u}_{k+j|k}\right) + \sum_{j=N}^{H-1} \ell\left(e_{k+j|k}, 0\right)}^{period\ 1} \quad (20) \\ &+ \overbrace{\sum_{j=H}^{T_f-1} \ell\left(e_{k+j|k}, 0\right) + F\left(x_{k+T_f|k}\right)}^{period\ 2} = \sum_{j=0}^{H-1} \ell\left(e_{k+j|k}, \bar{u}_{k+j|k}\right) + F\left(x_{k+H|k}\right) \end{aligned}$$

where it must be noticed that $\bar{u}^i_{k+j|k} = 0$ for $j=N,...,H-1$.

b) For $k= T_f-N,...,T_f-1$ (that is, $H\leq N$), cost (20) becomes

$$V_k = \overbrace{\sum_{j=0}^{N-1} \ell\left(e_{k+j|k}, \bar{u}_{k+j|k}\right)}^{period\ 1} + \overbrace{F\left(x_{k+N|k}\right)}^{period\ 2} . \quad (21)$$

Remark 9: In Theorem 1 we state some convergence results of the Lyapunov function defined by the IHMPC strategy. In addition, these concepts can be extended to determine a *variability index* in order to establish a quantitative concept of stability (β -stability), as it was highlighted by Srinivasan and Bonvin, 2007. To make this extension, the MPC stability conditions (rather than convergence conditions) must be defined following the stability results of (Scokaert et al., 1997). An extension of this remark can be seen in the Appendix A.

4. ILUSTRATIVE EXAMPLE

Example 1. In order to evaluate the proposed controller performance, we consider first a linear system (Lee and Lee, 1997) given by $G(s)=1/15s^2+8s+1$. The MPC parameters were tuned as $Q = 1500$, $R = 0.5$ and $T = 1$. Figure 2 shows the obtained performance in the controlled variable where the difference with the reference is undistinguished. Given that the problem assumes that no information about the input reference is available, the input sequence $\boldsymbol{u}$ and $\bar{\boldsymbol{u}}$ are equals.

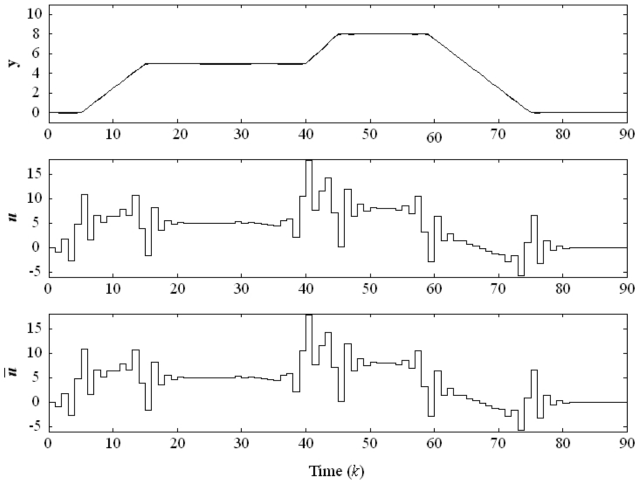

Fig. 2. Reference, output response according to the input variables $\boldsymbol{u}$ and $\bar{\boldsymbol{u}}$.

The MPC cost function is showed in Fig. 3. According to the proof of Theorem 1 (nominal case), this cost function is monotonically decreasing.

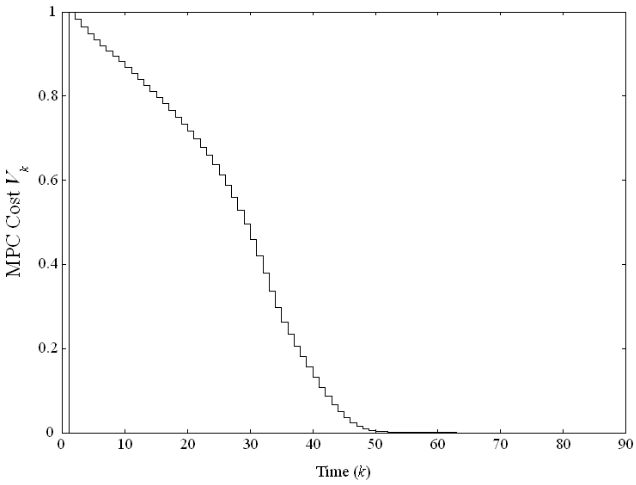

Fig. 3. Normalized MPC cost function. Here, the normalized cost function is obtained as V_k / V_k^{max} .

Example 2. Consider now a nonlinear-batch reactor where an exothermic and irreversible chemical reaction takes place, (Lee and Lee, 1997). The idea is to control the reactor temperature by manipulating the inlet coolant temperature. Furthermore, the manipulated variable has minimum and maximum constrains given by: $Tc_{min} \le Tc \le Tc_{max}$, where Tc_{min} = -25ºC, Tc_{max} = 25ºC and, Tc is written in deviation variable. In addition, to show how the MPC controller works, it is assumed that a previous information about the cooling jacked temperature ($u = Tc$) is available.

Here the proposed MPC was implemented and the MPC parameters were tuned as, Q = 1000 , R =5 and T = 1 min. The nominal linear model used for predictions is the same proposed by Adam (2007).

Figure 4 shows both the reference and the temperature of the batch reactor are expressed in deviation variable. Furthermore, the manipulated variable and the correction made by the MPC, $\bar{\boldsymbol{u}}$ are shown.

Notice that, 1) the cooling jacked temperature reaches the maximum value and as a consequence the input constraints becomes active in the time interval from 41 minutes to 46 minutes; 2) similarly, when the cooling jacked temperature reaches the minimum value, the other constraint becomes active in the time interval from 72 minutes to 73 minutes; 3) the performance is quite satisfactory in spite of the problem is considerably nonlinear and, 4) given that it is assumed that a previous information about the cooling jacked temperature is available, the correction $\bar{\boldsymbol{u}}$ is somewhat smaller than $\boldsymbol{u}$ (Fig. 4).

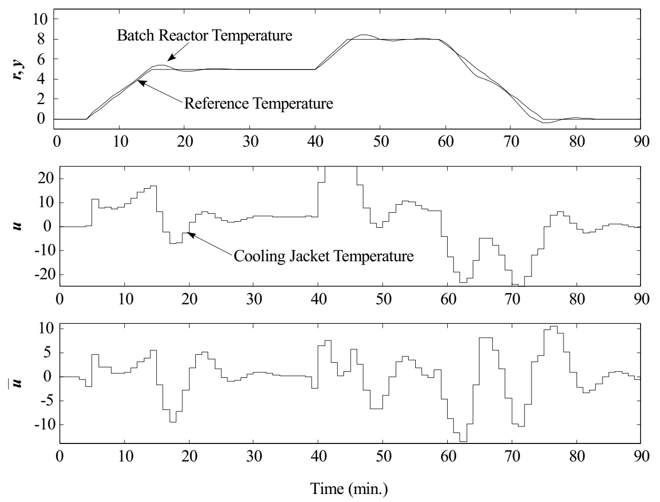

Fig. 4. Temperature reference and controlled temperature of the batch reactor. Also, the cooling jacked temperature ($\boldsymbol{u}$) and the correction ($\bar{\boldsymbol{u}}$) are showed.

5. CONCLUSIONS

In this paper a different formulation of a stable IHMPC for batch processes is presented. The main idea is to consider a virtual infinite horizon for predictions, in order to guarantee that the cost decreases from a given instant to the next one. For the case in which the process parameters substantially change from one batch to the next (there is not an exact repetition), simulation examples show that the controller achieves the expected performance. More important, for the case in which the system parameters remain unmodified for several batch runs, the formulation can be straightforwardly extended to an MPC with learning properties, which updates the underling control of the closed-loop paradigm to achieve nominal perfect control performance. This property is developed in the second part of this work (González et al., 2009).

APPENDIX A

In Theorem 1 we state some convergence results of the Lyapunov function defined by the IHMPC strategy. Now, these concepts are extended to determine a *variability index* in order to establish a quantitative concept of stability (β -stability). First, we will recall the following exponential stability results.

Theorem 2 (Scokaert et al., 1997):

Let assume for simplicity that a state reference, x_k^r, is provided, such that $y_{k+j}^r = Cx_{k+j}^r, \quad j = 0,\cdots,T_f$, and no disturbance is present . If there exist constants a_x, a_u, b_u, c_x, c_u and d_x such that

$$\underline{\gamma}\cdot\|x\|^\sigma \le \ell(x,u) = \|x\|_Q^2 + \|u\|_R^2 \le c_x\cdot\|x\|^\sigma + c_u\cdot\|u\|^\sigma \qquad \text{(A1)}$$

$$\left\|u_{k+j|k}^{i^{opt}}\right\| \le b_u\|x_k\|, \text{ for } j=0,\dots,T_f\text{-}1 \qquad \text{(A2)}$$

$$\|Ax+Bu\| \le a_x\|x\| + a_u\|u\| \qquad \text{(A3)}$$

$$F(x) \le d_x\|x\|^\sigma \qquad \text{(A4)}$$

then

$$\underline{\gamma}\cdot\|x_k\|^\sigma \le V_k^{i^{opt}}(x_k) \le \overline{\gamma}\cdot\|x_k\|^\sigma, \qquad \text{(A5)}$$

$$V_k^{i^{opt}}(x_k) \le -\underline{\gamma}\cdot\|x_k\|^\sigma \qquad \text{(A6)}$$

with $\quad \overline{\gamma} = \left(c_x\cdot\sum_{i=0}^{N-1}\alpha_i^\sigma + N\cdot c_u\cdot b_u^\sigma + d_x\cdot\alpha_N^\sigma\right),$

$\alpha_j = a_x\cdot\alpha_{j-1} + a_u\cdot b_u$ and $\alpha_0 = a_x + a_u\cdot b_u$.

Proof. The proof of this theorem can be found in Scokaert et al., 1997.

Condition (A1) is easy to determine in terms of the eigenvalues of matrices Q and R. Condition (A2), which are related to the Lipschitz continuity of the input, holds true under certain regularity conditions of the optimization problem.

Now, we define the following variability index, similar to the one presented in Srinivasan and Bonvin (2007):

$$\xi_V = \max_{V_0^{i^{opt}}=\delta}\left(\frac{\sum_{k=0}^{T_f-1}V_k^{i^{opt}}}{V_0^{i^{opt}}}\right).$$

With the last definition, the concept of β -stability for finite-duration systems is as follows.

Definition (Srinivasan and Bonvin, 2007)**:** The closed-loop system obtained with the proposed IHMPC controller is intra-trial β-stable around the state trajectory x_k^r if there exists $\delta > 0$ such that $\xi_V \le \beta$.

Now, the following result can be stated.

Theorem 3 (quantitative β-stability)

Let assume for simplicity that a state reference, x_k^r, is provided, such that $y_{k+j}^r = Cx_{k+j}^r, \quad j = 0, \cdots, T_f$, and no disturbance is present. If there exist constants a_x, a_u, b_u, c_x, c_u and d_x as in Theorem 3, then, the closed-loop system obtained with system(5)-(7) and the proposed IHMPC controller law is intra-trial β-stable around the state trajectory x_k^r, with $\beta = \frac{\left[\overline{\gamma} + \sum_{n=1}^{T_f - 1}\left(\overline{\gamma} - \underline{\gamma}\right)^n\right]}{\underline{\gamma}}$.

Proof. from the recursive use of (17), together with (A1), (A5) and (A6), we have

$$V_{k+1}^{i^{opt}} \le V_k^{i^{opt}} - \ell\left(\overline{x}_k^i, \overline{u}_k^i\right) \le \overline{\gamma} \cdot \left\|\overline{x}_k^i\right\|^{\sigma} - \underline{\gamma} \cdot \left\|\overline{x}_k^i\right\|^{\sigma} = \left(\overline{\gamma} - \underline{\gamma}\right)\left\|\overline{x}_k^i\right\|^{\sigma},$$

for $k = 0, \cdots, T_f - 2$

So, we can write:

$$\sum_{k=0}^{T_f - 1} V_k^{i^{opt}} \le \left[r + \sum_{n=1}^{T_f - 1}\left(\overline{\gamma} - \underline{\gamma}\right)^n\right] \cdot \left\|\overline{x}_0^i\right\|^{\sigma}.$$

Therefore,

$$\frac{\sum_{k=0}^{T_f - 1} V_k^{i^{opt}}}{V_0^{i^{opt}}} \le \frac{\left[\overline{\gamma} + \sum_{n=1}^{T_f - 1}\left(\overline{\gamma} - \underline{\gamma}\right)^n\right]}{\underline{\gamma}}$$, since $\underline{\gamma} \cdot \left\|x_0\right\|^{\sigma}$ is a lower bound of $V_0^{i^{opt}}$ (that is, $\underline{\gamma} \cdot \left\|x_0\right\|^{\sigma} \le V_0^{i^{opt}}$).

Finally, $\beta = \frac{\left[\overline{\gamma} + \sum_{n=1}^{T_f - 1}\left(\overline{\gamma} - \underline{\gamma}\right)^n\right]}{\underline{\gamma}}$.□

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