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Application of IHMPC to an unstable reactor system: study of feasibility and performance

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Abstract: Almost all the theoretical aspects of Model Predictive Control (MPC), such as stability, recursive feasibility and even the optimality are now well established for both, the nominal and the robust case. The stability and recursive feasibility are usually guaranteed by means of additional terminal constraints, while the optimality is achieved considering closed-loop predictions. However, these significant improvements are not always applicable to real processes. An interesting case is the control of open-loop unstable reactor systems. There, the traditional infinite horizon MPC (IHMPC), which constitutes the simplest strategy ensuring stability, needs to include an additional terminal constraint to cancel the unstable modes, producing in this way feasibility problems. The terminal constraint could be an equality or an inclusion constraint, depending on the local controller assumed for predictions. In both cases, however, a prohibitive length of the control horizon is necessary to produce a reasonable domain of attraction for real applications. In this work, we propose an IHMPC formulation that has maximal domain of attraction (i.e. the domain of attraction is determined by the system and the constraints, and not by the controller) and is suitable for real applications in the sense that it accounts for the case of output tracking, it is offset free if the output target is reachable, and minimizes the offset if some of the constraints become active at steady-state.

Keywords: model predictive control, domain of attraction, unstable reactor.

1. INTRODUCTION

When a constrained open-loop unstable system, as an unstable reactor system, is attempted to be controlled the guarantee of recursive feasibility and constrained stability is a highly desirable controller property. First, the maximal stabilizable sets associated to the system equilibrium should be carefully determined since, opposite to what happens with stable systems, input constraints could make impossible the rejection of large disturbances, independently of the controller. Then, a controller with guaranteed stability that explicitly takes into account these limitations should be designed. In this context, MPC appears to be the most suitable option. In fact, the stability, feasibility and even optimality of MPC is now well established in the theoretical aspects (Mayne et al. (2000), Rawlings and Mayne (2009)). Standard approaches use the dual-mode prediction paradigm (Scokaert and Rawlings (1998)) in conjunction with an infinite horizon. Within this paradigm it is assumed that a fixed unconstrained feedback K (local controller) proceeds for predictions beyond the control horizon. stabilizing in this way the unstable modes. In this context, a major obstacle is to establish a trade-off between the desirable volume of the domain of attraction (the set of states for which the controller can generate a feasible input), the overall complexity (computational cost), and the achievable performance for a given control horizon (degree of optimality). Assuming that the control horizon is chosen small for computational reasons, the domain of attraction is dominated by the aggressiveness of the fixed unconstrained feedback, since additional (terminal) constraints are needed to assure the feasibility of the local control law. For stable systems, the null local controller K=0 (Rawlings and Muske (1993)), which represents the poorest tuned local controller, produces the maximal domain of attraction. This is the case of the classical infinite horizon MPC (IHMPC). However, for the general case of unstable systems, a terminal constraint that cancels the unstable modes is needed (as they cannot be steered to the origin by the proposed local null controller), producing again a severe reduction of the resulting domain of attraction. Nagrath et al. (2002) presented an interesting example of a open-loop unstable jacketed chemical reactor (CSTR) controlled by MPC. The objective is to keep the temperature of the reacting mixture constant at a desired value, while the disturbances to the system include the feed temperature and the jacket feed temperature and the only manipulated variable is the jacket flow rate. For this kind of system, it is usual to adopt a cascade control structure: an inner-loop for the control of the jacket temperature and an outer-loop for the control of the reactor temperature. The authors compared three different controllers: a classical cascade controller, a finite horizon MPC and an IHMPC. The output performance is clearly better for the MPC strategies, while only the IHMPC assures stability. As the authors declared, the problem with the IHMPC is that to implement appropriately the cancellation of the unstable modes (terminal constraint) a prohibitive large control horizon is needed. Again, the problem of a reduced domain of attraction for small control horizons arises.

More recently, González and Odloak (2009) reformulates the original IHMPC to allow an augmented domain of attraction for both, stable and unstable systems. The main idea was to include an appropriate set of slack variables into the MPC optimization problem that, together with some model formulation properties, allows a maximal closed-loop domain of attraction (i.e., the domain of attraction is determined by the system and the constraints, and not by the controller).

This paper proposes the study of the application of IHMPC to an unstable reactor system in terms of the domain of attraction and the output performance. Furthermore, a generalization of the IHMPC proposed in González and Odloak (2009) is presented, and some steady state optimality properties, related to the capability of the controller to account for unreachable output references, are discussed.

Notation: vector (a,b) denotes $\left[a^T b^T\right]^T$; for a given λ, $\lambda X = \{\lambda x : x \in X\}$. For a symmetric positive definite matrix P, $\|x\|_P = \sqrt{x^T P x}$ denotes the weighted Euclidean norm. Matrix $I_n \in \mathbb{R}^{n\times n}$ denotes the identity matrix. $0_{n,m} \in \mathbb{R}^{n\times m}$ denotes the null matrix. Consider $a \in \mathbb{R}^{na}$, $b \in \mathbb{R}^{nb}$ and a set $\Gamma \subset \mathbb{R}^{na+nb}$, then the projection operation is defined as $Proj_a(\Gamma) = \left\{a \in \mathbb{R}^{na} : \exists b \in \mathbb{R}^{nb}, (a,b) \in \Gamma\right\}$. Given two sets S_1, S_2, the set $S_1 \backslash S_2$ is defined as $S_1 \backslash S_2 = \{x : x \in S_1 \text{ and } x \notin S_2\}$.

2. REACTOR SYSTEM

Consider a continuous stirred-tank reactor (CSTR), in which an exothermic adiabitic irreversible first-order reaction (A $\rightarrow$ B) is described by the following non-linear state equations (Nagrath et al. (2002), Russo and Bequette (1996)):

$$\dot{x}_1 = -[q + \phi\mathcal{K}(x_2)]x_1 + qx_{1f} \tag{1}$$

$$\dot{x}_2 = \beta\phi\mathcal{K}(x_2)\, x_1 - (q+\delta)x_2 + \delta x_3 + qx_{2f} \tag{2}$$

$$\dot{x}_3 = \delta_1\delta\delta_2 x_2 - \delta_1(q_c + \delta\delta_2)x_3 + \delta_1 q_c x_{3f} \tag{3}$$

In these equations x_1 is the dimensionless concentration of reactant A, x_2 is the dimensionless reactor temperature, x_3 is the dimensionless jacket temperature, x_{1f} is the dimensionless feed concentration of reactant A to the reactor, x_{2f} is the dimensionless feed temperature, x_{3f} is the dimensionless jacket feed temperature and q_c is the jacket flow rate. The state vector is defined as $x := [x_1\ x_2\ x_3]^T$, while x_{1f}, x_{2f} and x_{3f} are the possible disturbances to the reactor. In this case, x_{1f} is considered constant, and then the disturbance vector will be given by $l := [x_{2f}\ x_{3f}]^T$. Clearly, the set of equations presented before is nonlinear. Eq. (1) represents the dynamic material balance for the reactant A, Eq. (2) represents the dynamic energy balance inside the CSTR, and Eq. (3) represents the dynamic energy balance around the cooling jacket. For the three equations, the usual assumptions of constant volume, perfect mixing and constant physical parameters are made. The traditional operation of a CSTR consists in an indirect control of concentration by means of the control of the reactor temperature (x_2), using the cooling jacket flow rate (q_c) as manipulated variable. However, as Russo and Bequette (1996) and Russo and Bequette (1998) remark, a multiplicity behavior could be found for the three-state CSTR model when q_c is the manipulated variable. The existence of the input multiplicities in the system can severely degrade the performance of controlled output and moreover, unfeasible operation regions can appear. For this reason, the selection of an adequate operation point is a key for the correct operation of the reactor.

Several authors studied and derived conditions for steady-state multiplicities for a CSTR. The parameters chosen in this work are those shown in Russo and Bequette (1998), which exhibit open-loop unstable behavior for a range of reactor temperatures bounded by the limit points: $\phi = 0.072$, $\beta = 8.0$, $\delta = 0.3$, $\gamma = 20$, $q = 1.0$, $\delta_1 = 10$, $\delta_2 = 1.0$, $x_{1f} = 1.0$, $x_{2f} = 0.0$ and $x_{3f} = -1.0$.

3. MODELING THE SYSTEM AND CONSTRAINTS

The aim of this section is to discuss both, a linear prediction model and the way the constraints affect it when unstable systems are attempted to be described. As was shown in Nagrath et al. (2002) block diagonal transition matrices, which separate the unstable from the stable modes of the systems, are desirable to represent unstable systems for IHMPC. In this context, a suitable model is as follows:

$$\begin{bmatrix} x^i(k+1) \\ x^{un}(k+1) \\ x^{st}(k+1) \end{bmatrix} = \begin{bmatrix} I_{ni} & 0 & 0 \\ 0 & F^{un} & 0 \\ 0 & 0 & F^{st} \end{bmatrix} \begin{bmatrix} x^i(k) \\ x^{un}(k) \\ x^{st}(k) \end{bmatrix} + \begin{bmatrix} B^i \\ B^{un} \\ B^{st} \end{bmatrix} \Delta u(k) \tag{4}$$

$$y(k) = \begin{bmatrix} \Upsilon^i & \Upsilon^{un} & \Upsilon^{st} \end{bmatrix} \begin{bmatrix} x^i(k) \\ x^{un}(k) \\ x^{st}(k) \end{bmatrix},$$

where $x^i \in X^i \subseteq \mathbb{R}^{ni}$, $ni = max(nu, ny)$, represents the integrating modes artificially induced by the incremental form of the model, $x^{st} \in X^{st} \subseteq \mathbb{R}^{ns}$ represents the stable modes, $x^{un} \in X^{un} \subseteq \mathbb{R}^{nun}$ represents the original unstable modes of the system, $\Delta u(k) = u(k) - u(k-1) \in \mathbb{R}^{nu}$, $y(k) \in \mathbb{R}^{ny}$. This particular block diagonal form of model (4) can be obtained from the step response of the transfer function model (González and Odloak (2009), Rodrigues and Odloak (2003)), or by an appropriate similarity transformation of a given state space model. The system has nun and ns unstable and stables poles, respectively. In addition, ni is the number of integrating modes (introduced by the incremental form of the model) that is equal to the maximum between the number of system input and outputs. Matrix Υ^i is given by $\Upsilon^i = \left[I_{ni} 0_{ni,(ni-ny)}\right]$ if $nu > ny$, and by $\Upsilon^i = I_{ni}$ if $nu \leq ny$. Matrices Υ^{un} and Υ^{st} account for the effect of the unstable and stable states in the output.

On the other hand, the input feasible set U is defined as follows:

$$U = \{\Delta u : -\Delta u_{max} \leq \Delta u \leq \Delta u_{max} \text{ and } u_{min} \leq u(k-1) \leq u_{max}\},$$

where $u(k-1)$ is the past value of the input u. In addition, it is assumed that the states $x = \left(x^i, x^{un}, x^{st}\right)$ are constrained to belong to a set X, given by $X = X^i \times X^{un} \times X^{st}$. Here, this set is defined by the operating window of the process. Set X_i must satisfy the input constraints, as follows: $B^i u_{min} \leq x^i \leq B^i u_{max}$.

Remark 1. Notice that the present formulation takes a special care of input increments. Firstly, the input increment is considered as input in the system (4), instead of the input itself. This property gives an alternative way to achieve an offset-free control, in comparison to the target calculation strategy used in Nagrath et al. (2002), that needs a separate optimization problem to be solved. Secondly, input increment constraints are included together with the input constraints. This constitutes an important feature of real processes usually disregarded by the typical MPC formulations, and contributes to a better description of the whole system, since this kind of constraints limits the maximal domain of attraction of the system, as can be seen in the next sub-section.

3.1 Characterization of the constrained steady states of the system

Without loss of generality it is assumed here that $nu = ny = ni$ (for non-square systems the procedure to characterize the steady state is similar). For a given output target (or set-point) y^{sp}, any steady state of the system $\left(x_s^i, x_s^{un}, x_s^{st}, \Delta u_s\right)$ associated with this target output should satisfy the following condition

$$\begin{bmatrix} \begin{bmatrix} I_{ni} - I_{ni} & 0 & 0 \\ 0 & F^{un} - I_{nun} & 0 \\ 0 & 0 & F^{st} - I_{ns} \end{bmatrix} & \begin{bmatrix} B^i \\ B^{un} \\ B^{st} \end{bmatrix} \\ \left[\Upsilon^i \ \Upsilon^{un} \ \Upsilon^{st}\right] & 0 \end{bmatrix} \begin{bmatrix} x_s^i \\ x_s^{un} \\ x_s^{st} \\ \Delta u_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ y^{sp} \end{bmatrix}. \quad (5)$$

From equation (5), and assuming that $rank\left(B^i\right) = nu$, it follows that any steady state of the system are given by $\left(x_s^i, x_s^{un}, x_s^{st}, \Delta u_s\right) = (y^{sp}, 0, 0, 0)$. This last condition represents a useful property of the specific model formulation (4), as it says that any possible steady state can be condensed in a single state component $\left(x_s^i\right)$, while the other states are null [1].

Now, since the system is subject to constraints, it should be steered to those steady states that satisfy the constraints. The set of these admissible steady states is defined as [2]

$$X_s = \left\{x_s = \left(x_s^i, x_s^{un}, x_s^{st}\right) \in X : x_s^i \in X^i, \ x_s^{un} \in \{0\} \text{ and } x_s^{st} \in \{0\}\right\}.$$

Notice that from the latter definition, it follows that the set of feasible output steady states, $y_s = Cx_s = x_s^i$, is directly given by X^i (this means that any feasible integrating state is a steady state of the system).

Remark 2. It is frequent in real applications that the desired output set-point y^{sp} is not reachable, that is, the desired steady state $x_s = (y^{sp}, 0, 0)$ is not in X_s. This could be caused by a mismatch between the model used to compute the optimal steady state and the model used to compute the dynamic forecast. Despite in this case an output offset will necessarily appear, this should not be a cause of instability.

3.2 Characterization of the stabilizable sets for the non-stable states

A remarkable characteristic of the constrained unstable systems is that there exists a limited set of non-stable states, called maximal stabilizable set, out of which the system cannot be stabilized by any controller. It is useful at this point to group the integrating and the pure unstable modes into a single vector, $x^{nst} = \left(x^i, x^{un}\right)$, of non-stable states. In this way system (4) can be rewritten as

$$\begin{bmatrix} x^{nst}(k+1) \\ x^{st}(k+1) \end{bmatrix} = \begin{bmatrix} F^{nst} & 0 \\ 0 & F^{st} \end{bmatrix} \begin{bmatrix} x^{nst}(k) \\ x^{st}(k) \end{bmatrix} + \begin{bmatrix} B^{nst} \\ B^{st} \end{bmatrix} \Delta u(k)$$

$$y(k) = \left[\Upsilon^{nst} \ \Upsilon^{st}\right] \begin{bmatrix} x^{nst}(k) \\ x^{st}(k) \end{bmatrix},$$

where $B^{nst} = \left[B^{i^T} \ B^{un^T}\right]^T$, $F^{nst} = diag\left(I_{ni}, F^{un}\right)$, $\Upsilon^{nst} = \left[\Upsilon^i \ \Upsilon^{un}\right]$. Now we exploit the steady state characterization presented in the last section in order to define some useful sets. First, let us consider the (equilibrium) set of feasible non-stable steady states, $X_s^{nst} = Proj_{x^{nst}}(X_s)$. Based on this definition, it is possible to define the set of non-stable states that can be admissibly steered, by means of an admissible sequence of j control actions, from $X^{nst} = X^i \times X^{un}$ to the equilibrium set X_s^{nst}:

$$St_j^{nst}\left(X^{nst}, X_s^{nst}\right) = \{x^{nst}(0) \in X^{nst} : \text{ for all } k = 0, \cdots, j-1, \exists \Delta u(k) \in U \text{ such that } x^{nst}(k) \in X^{nst} \text{ and } x^{nst}(j) \in X_s^{nst}\}.$$

This set (called the stabilizable set for the non-stable states) is a control invariant set for states x^{nst}, for all $j \geq 1$ (see Lemma 1 in González and Odloak (2009)).

Remark 3. As can be seen in Remark 1 of González and Odloak (2009), the set $St_j^{nst}\left(X^{nst}, X_s^{nst}\right)$ tends to a limited set as the number of steps j tends to ∞. So, by increasing the index j up to N, in such a way that $St_{N+1}^{nst}\left(X^{nst}, X_s^{nst}\right) \approx St_N^{nst}\left(X^{nst}, X_s^{nst}\right)$, it is possible to

[1] Notice that the steady state is condensed in the state component that was artificially included in model (4)

[2] Notice that, as was already said, X^i is such that the input constraints are fulfilled (i.e., $B^i u_{min} \leq x^i \leq B^i u_{max}$)

define the *largest possible domain of attraction* for the non-stable states as $\Theta^{nst} = St_N^{nst}(X^{nst}, X_s^{nst})$.

Remark 4. If we now define $\Theta = \{x \in X : x^{nst} \in \Theta^{nst}\}$, then this set is the maximal domain of attraction of any controller (*largest possible domain of attraction* of the system) since it does not depend on the selected control law, but on the nature of the system and the states, as well as on the input constraints.

Remark 5. The equilibrium set for the pure unstable states is given by $X_s^{un} = Proj_{x^{un}}(X_s) = \{0\}$. So, it is possible to define the stabilizable set for the unstable states as $St_j^{un}(X^{un}, \{0\})$. Furthermore, the *largest possible domain of attraction* for the unstable states is given by $\Theta^{un} = Proj_{x^{un}}(\Theta^{nst})$.

4. IHMPC FORMULATIONS

4.1 Classical IHMPC

As was established in Nagrath et al. (2002), the main advantage of the infinite over the finite horizon is the elimination of the requirement of tuning for nominal stability. For unstable systems, however, the non-stable modes must be canceled to achieve constrained stabilizability of the predictions (a constraint is included that forces this cancellation at the end of the control horizon m). For tracking a non-zero target, the classical IHMPC formulation adapted to model (4) is as follows:

Problem 1

$$\min_{\Delta u_k} V_k = \sum_{j=0}^{m-1} \Big\{ \| x(k+j|k) - x^{sp} \|_Q^2 + \| \Delta u(k+j|k) \|_R^2 \Big\} + \left\| x^{st}(k+m|k) \right\|_P^2$$

subject to:

$$\Delta u(k+j|k) \in U, \quad j = 0, \cdots, m-1$$

$$x^i(k+m|k) - y^{sp} = 0 \tag{6}$$

$$x^{un}(k+m|k) = 0 \tag{7}$$

where $x^{sp} = (y^{sp}, 0, 0)$, $x(k|k) = x(k)$, $\Delta u(k+j|k)$ is the control move computed at time k to be applied at time $k+j$, m is the control horizon, Q and R are positive weighting matrices of appropriate dimension, y^{sp} is the output reference, and $\Delta u_k = (\Delta u(k|k), \cdots, \Delta u(k+m-1|k))$. Because of the terminal constraints (6) and (7), the cost of the IHMPC can be written as a finite horizon cost with a terminal penalty term (the terminal matrix P is computed by solving the Lyapunov equation $P = Q + F^{st^T} P F^{st}$).

Two main problems arise when this formulation is intended to be applied. The domain of attraction for the non-stable modes is given by the m-stabilizable set $St_m^{nst}(X^{nst}, \{0\})$, because of constraints (6) and (7). This set is very small, mainly if the practical case of input increment constraints is considered, because it considers as target set the origin (instead of the equilibrium set X_s^{nst}) and because it considers only m steps to reach this target set. On the other hand, if an unreachable set-point is used, the cost becomes unbounded and feasibility/stability is lost.

4.2 IHMPC with large domain of attraction

A possible solution to these problems is the inclusion of slack variables to relax the terminal constraints. In González and Odloak (2009) a formulation was presented that accounts for the problem of including slack variables into the infinite horizon MPC optimization problem, with the final objective of enlarging the resulting domain of attraction. The main idea of this approach was to separate the convergence (main objective of an IHMPC) in two steps: the first one is devoted to steer the non-stable state x^{nst} to the set $St_m^{nst}(X^{nst}, X_s^{nst})$ in a finite number of steps; while the second one is devoted to steer the integrating and stable states to the desired equilibrium point $\big((x^i, x^{st}) \to (y^{sp}, 0)\big)$. The slack variables assure the feasibility at any time and for any non-stable state in the maximal domain of attraction $St_\infty^{nst}(X^{nst}, X_s^{nst})$. As a result, the controller has the largest possible domain of attraction, i.e. the domain of attraction is given by Θ, which does not depend on the control law.

4.3 A novel IHMPC with large domain of attraction

The strategy presented in González and Odloak (2009) has some limitations, since it does not deal with an arbitrary number of unstable states. A possible generalization of this strategy consists of using a *generalized Minkowski functional*, associated to the stabilizable sets $St_j^{un}(X^{un}, \{0\})$, for $m \le j \le N$ (where N is as in Remark 3), as the cost function of a first optimization problem of a two-stage MPC formulation. Before presenting the novel IHMPC formulation, we introduce the following definitions:

Definition 1. Given a convex set $S \subset X$, the Minkowski functional Ψ_S associated to S is defined as

$$\Psi_S(x) = \inf\{\mu \ge 0 : x \in \mu S\}.$$

To see the properties of the function $\Psi_S(x)$, see Blanchini (1999).

Consider now a sequence of 0-symmetric convex sets $S_1 \subset S_2 \subset \ldots \subset S_n$. Then, a corresponding sequence of pairwise disjoint sets, $\Upsilon_{i+1} = \text{int}(S_{i+1}) \backslash \text{int}(S_i)$, $i = 1, \ldots, n-1$, which are a partition of S_n, could be defined. Now, following the idea of the Minkowski functional presented above, it is possible to associate a function to the whole sequence, in such a way that the level surfaces of this new functional are the contours of the sets S_i:

Definition 2. Given a sequence of convex sets $S_1 \subset S_2 \subset \ldots \subset S_n$, the generalized Minkowski functional $\Psi_{\{S_1,\ldots,S_n\}}$ associated to S_i is defined as

$$\Psi_{\{S_1,\ldots,S_n\}}(x) = \begin{cases} \Psi_{\nu_i S_i}(x) & \text{if } \ x \in \Upsilon_{i+1}, \ i = 1, \ldots, n-1 \\ \Psi_{\nu_1 S_1}(x) & \text{if } \ x \in S_1 \end{cases}$$

where the coefficients ν_i are such that $\nu_i S_i \supseteq \nu_{i+1} S_{i+1}$.

Remark 6. To obtain the coefficients ν_i the following optimization problem must be solved:

$$\nu_i = \min\{\nu \ge 0 : \nu S_i \supseteq \nu_{i+1} S_{i+1}\},$$
$$\text{for } \ i = 1, \ldots, n-1, \ \text{ and } \ \nu_n = 1.$$

Once the coefficient are obtained, the sequence of sets $\nu_i S_i$ are such that $\nu_i S_i \supset \nu_{i+1} S_{i+1}$, for $i = 1, \ldots, n-1$.

The generalized Minkowski functional applied to unstable stabilizable sets has the following properties, which are useful for the MPC formulation: $\Psi_{\{St_m^{un},\cdots,St_N^{un}\}}(x^{un})$ is greater than zero for all non-null values of x^{un}, $\Psi_{\{St_m^{un},\cdots,St_N^{un}\}}(x^{un})$ is null if and only if x^{un} is null, and the contour of the stabilizable sets $\{St_m^{un},\cdots St_N^{un}\}$ are level surfaces of the functional $\Psi_{\{St_m^{un},\cdots,St_N^{un}\}}(x^{un})$. Based on the above definitions, the proposed IHMPC formulation, which consists of the following optimization problems that are solved sequentially:

Problem 2a

$$\min_{\Delta u_{a,k}} V_{a,k} = \Psi_{\{St_m^{un},\cdots,St_N^{un}\}}\left(x^{un}(k+m|k)\right)$$

subject to:

$$\Delta u_a(k+j|k) \in U, \quad j=0,\cdots,m-1$$

Problem 2b

$$\min_{\Delta u_{b,k},\delta_k^i,\delta_k^{un}} V_{b,k} = \sum_{j=0}^{m-1}\left\{\| x(k+j|k) - x^{sp} + \delta(k,j)\|_Q^2 + \|\Delta u_b(k+j|k)\|_R^2\right\} + \left\|x^{st}(k+m|k)\right\|_P^2 + \left\|\delta_k^i\right\|_S^2$$

subject to:

$$\Delta u_b(k+j|k) \in U, \quad j=0,\cdots,m-1$$

$$x^i(k+m|k) - y^{sp} + \delta_k^i = 0 \tag{8}$$

$$x^{un}(k+m|k) + F^{un^m}\delta_k^{un} = 0 \tag{9}$$

$$\Psi_{\{St_m^{un},\cdots,St_N^{un}\}}\left(x^{un}(k+m|k)\right) \le \Psi^* \tag{10}$$

where S is a positive weighting matrices of appropriate dimension, $\delta(k,j) = \left(\delta_k^i, F^{un^j}\delta_k^{un}, 0\right)$ are slack variables, and Ψ^* is the optimal cost of Problem 2a. Because of the slacked terminal constraints (8) and (9), the cost of this IHMPC can be written as a finite horizon cost with a terminal penalty term, as it is done in a classical IHMPC. Constraint (10) forces the unstable states to remain in the stabilizable set determined by Problem 2a.

The following algorithm produces a stabilizing control law with a domain of attraction given by Θ:

Algorithm 1. Solve Problem 2a and pass the optimal value of the cost to Problem 2b. Implement the first control action $\Delta u_b(k|k)$.

Also, the control sequence obtained from the execution of Algorithm 1 at successive time steps drives the output of the closed loop system asymptotically to a point that minimizes $\left\|\delta_k^i\right\|_S^2$ (particularly, if the output set-point is reachable, the output of the closed loop system is asymptotically steered to it without offset).

Remark 7. The terminal constraint for the unstable and integrating states in Problem 2b can be written as

$$\begin{bmatrix} C^i \\ C^{un} \end{bmatrix} \Delta u_{b,k} = \begin{bmatrix} y^{sp} - x^i(k) - \delta_k^i \\ -F^{un^m}\left(x^{un}(k) - \delta_k^{un}\right) \end{bmatrix},$$

where $C^i = \left[B^i\ B^i\ \cdots\ B^i\right]$, $C^{un} = \left[F^{un^{m-1}}B^{un}\ \ F^{un^{m-2}}B^{un}\ \cdots\ B^{un}\right]$. So, with this two-stage formulation, the control horizon m should be large enough to assure that matrix $\left[C^{i^T}\ C^{un^T}\right]^T$ is full rank.

5. APPLICATION TO THE UNSTABLE REACTOR

The aim of the simulation results is to compare the performance and feasibility of the proposed formulation with the classical IHMPC, when the unstable reactor of section 2 is considered. The objective is to control one output variable (reactor temperature) manipulating one input variable (jacket flow rate). Using Taylor series expansion and a convenient transformation, the following diagonalized discrete time linear model is obtained (T=0.05):

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1.339 & 0 & 0 \\ 0 & 0 & 0.9531 & 0 \\ 0 & 0 & 0 & 0.6167 \end{bmatrix}, \qquad B = \begin{bmatrix} 2.2361 \\ -0.2813 \\ -0.1931 \\ 1.2699 \end{bmatrix}$$

and

$$C = \left[\, -0.1977\ -0.9627\ -0.7735\ 0.0226 \,\right]$$

The input constraints are given by: $u_{\max} = 1.32$, $u_{\min} = 0$, $\Delta u_{\max} = 0.1$. The input increment bounds are chosen small to clearly show its effect on the controllers domain of attraction. The tuning parameters of the proposed IHMPC are: $Q = 50$, $R = 0.1$ and $S^i = 5\ 10^4$. The coefficients ν_i of the generalized Minkowski functional used for Problem 2a are given by: $\nu_5 = 1.8$, $\nu_6 = 1.5$, $\nu_7 = 1.4$, $\nu_8 = 1.3$, $\nu_9 = 1.15$ and $\nu_{10} = 1$. The tuning parameters of the classical IHMPC are: $Q = 50$,$R = 0.1$.

First, we analyze the domain of attraction for the non-stable modes of both controllers. As was already said, the domain of attraction of the IHMPC depends on the control horizon, and it is given by $St_m^{nst}(X^{nst},\{0\})$. Figure 1 shows these sets for $m = 10$, $m = 20$, $m = 24$, $m = 28$ and $m = 30$ (dashed-line). On the other hand, the domain of attraction of the proposed controller is the maximal domain of attraction of the system (and so, it does not depend on the control horizon), and it is given by $St_{15}^{nst}(X^{nst}, X_s^{nst})$ (solid-line). Notice that for $m > 28$ the domain of attraction of the IHMPC remains almost the same as the one obtained with $m = 28$ and furthermore, this maximal set is smaller than the maximal domain of attraction of the system. This shows that if input increment constraints are considered, the domain of attraction of the classical IHMPC could not be the largest possible, even for very large control horizons.

Now, the output performance of both controllers are compared. First, a state disturbance is simulated, such that the original non-stable states is in $St_7^{nst}(X^{nst}, X_s^{nst})$. The disturbance is given by: $x(0) = [0\ 0.12\ 0.2\ -0.2]^T$, which correspond to the following disturbance in the original state variables: $concentration\ of\ reactant = 0.12$, $reactor\ temperature = -0.23$ and $jacket\ temperature = -0.25$ (written as deviation variables). The proposed controller steers first the system to $St_m^{nst}(X^{nst}, X_s^{nst})$, with $m = 5$, in two steps, and then regulates the system to the desired equilibrium point. Figure 2 shows the evolution of the non-stable states (solid-line and circles). On the other hand, the classical IHMPC needs a control horizon equal

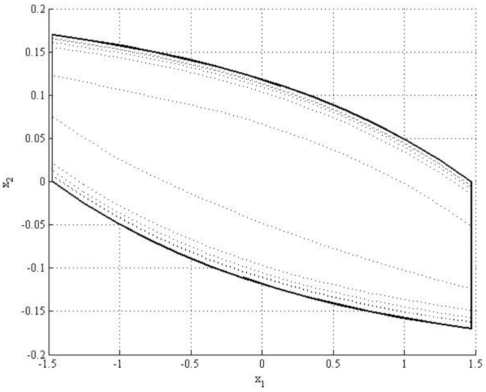

Fig. 1. Domain of attraction for both, the proposed controller and the classical IHMPC.

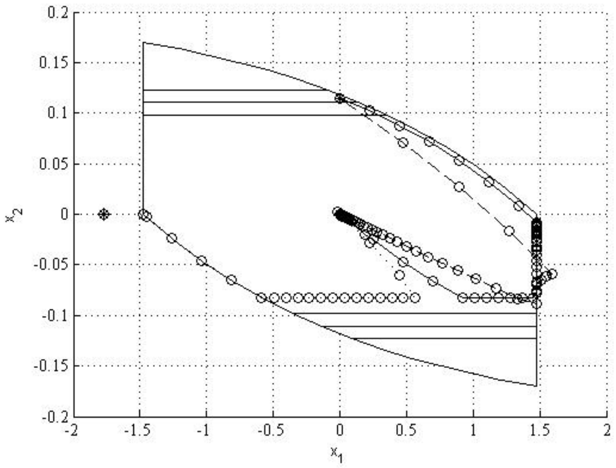

Fig. 2. Non-stable state evolution

to 30 to account for this disturbance. Figure 2 shows the non-stable state evolution for the IHMPC and $m = 15$ (dashed-line and circles), where it can be seen that the constraints are violated on the right hand side of the figure.

In a second stage, a set-point change of 0.1 is simulated. A control horizon of $m = 5$ was selected for the proposed controller, while a control horizon of $m = 9$ and $m = 12$ was selected for the IHMPC in order to make it feasible. Figure 3 shows the time responses, where it can be seen that the proposed controller has a slightly better performance. Finally, a set-point change of 0.35 is simulated, which is an unreachable target (it corresponds to an unreachable equilibrium state $x^{sp} = [-1.77\ 0\ 0\ 0]^T$). In this case, the IHMPC cannot be used, while the proposed controller, because of the effect of the slack variables, steers the system to a feasible point that minimizes the distance from the desired set-point. Figure 2 shows the non-stable state evolution (dotted-line and circles), together with the unreachable equilibrium state (circle-star).

6. CONCLUSION

A study was developed of the capability of IHMPC to account for unstable reactor systems. Performance and feasibility (together with stability) was studied, and a

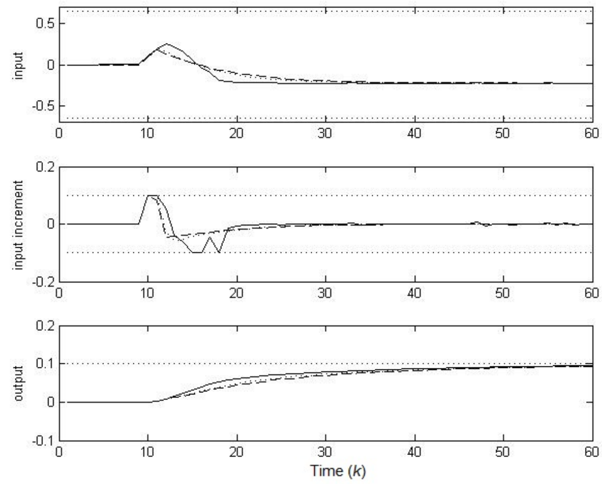

Fig. 3. Input, input increment and output responses for a set-point change.

new IHMPC formulation was presented that exploits the properties of a particular model structure and exhibits the largest possible domain of attraction. As other recent formulations, it guarantees recursive feasibility and stability for tracking both, reachable and unreachable output set-points. Furthermore, in the latter case, the resulting controller steers the system to an admissible stationary output that minimizes the distance to the desired set-point, without the necessity of a target calculation stage.

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