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# Stable MPC for tracking with maximal domain of attraction

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ABSTRACT

In this work, a stable MPC that maximizes the domain of attraction of the closed-loop system is proposed. The proposed approach is suitable to real applications in the sense that it accounts for the case of output tracking, it is offset free if the output target is reachable and minimizes the offset if some of the constraints are active at steady state. The new approach is based on the definition of a Minkowski functional related to the input and terminal constraints of the stable infinite horizon MPC. It is also shown that the domain of attraction is defined by the system model and the constraints, and it does not depend on the controller tuning parameters. The proposed controller is illustrated with small order examples of the control literature.



## 1. Introduction

The feasibility and/or stability of MPC for linear time invariant systems is now well established [1,2]. Standard approaches use the dual-mode prediction paradigm [3] in conjunction with an infinite horizon. Within this paradigm, it is assumed that a fixed unconstrained feedback $K$ (terminal controller) is used for predictions beyond the control horizon. In this case, a major obstacle is to establish a trade-off between the desirable volume of the domain of attraction (the set of states for which the controller can generate a feasible input), the overall complexity (computational cost), and the achievable performance for a given control horizon (degree of optimality). Assuming that the control horizon is chosen small for computational reasons, the domain of attraction is dominated by the fixed unconstrained feedback, since additional (terminal) constraints are needed to assure the feasibility of the local control law. Optimality of the dual approach can be obtained if the terminal linear controller is the Linear Quadratic Regulator (LQR) based on the same cost function as the MPC (same tuning weights). Many works have proposed strategies to enlarge the domain of attraction of this kind of controllers: In Refs. [4,5], the authors used a saturated local control law; in Ref. [6], the authors proposed a contractive terminal set given by a sequence of reachable sets. On the other hand, the extreme of a poorly tuned terminal controller is the null controller $K=0$ [7]. For unstable systems, however, this control strategy needs to include a terminal constraint that zeroes the unstable modes (as they cannot be steered to the origin by the proposed local null controller), reducing in this way the original terminal set, and so, the whole domain of attraction of the controller.

Another important point related to the feasibility/stability of linear MPC is the extension of the regulation problem to the tracking problem. In Ref. [8], the authors explain how a set-point change transforms the terminal set, producing in some cases a lack of feasibility. They also propose a general formulation to deal with the tracking problem, and in Ref. [9], they provide conditions for optimality of the controller. However, in some real applications (mainly in chemical processes), a reasonable lack of optimality is not so relevant if some steady-state conditions computed by a supervisory level are fulfilled [10]. On the other hand, an augmented domain of attraction is desirable to account for changes in the output targets and in the operating window of the process without loss of feasibility and/or stability. In this context [11], reformulates the original strategy based on the null terminal controller with terminal constraints to allow an augmented domain of attraction for both, stable and unstable systems.

The main idea of the approach proposed in Ref. [11] is to solve the MPC problem in two steps. In the first step the state is driven to a set from which the stable manifold can be reached at the end of the control horizon. Then, in the second step, the state is driven to the desired equilibrium point. A drawback of this formulation is that it is limited to the case of systems with at most one unstable mode per input, which may not be the case for important industrial systems [12,13]. This paper proposes an extension to the strategy presented in Ref. [11], introducing a more general way of dealing with the

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unstable modes. The original idea of using slack variables in the constraints that zero the unstable modes is extended by means of the use a functional based on the Minkowski functional. Furthermore, some steady state optimality properties, related to the capability of the controller to account for unreachable output references, are included.

*Notation*: For a symmetric positive definite matrix $P$, $||x||_P = \sqrt{x^TPx}$ denotes the weighted Euclidean norm. Matrix $I_n \in \mathbb{R}^{n\times n}$ denotes the identity matrix, and matrix $0_{n,m} \in \mathbb{R}^{n\times m}$ denotes the null matrix. Consider two sets $U$ and $V$, and a real number $\lambda$. The Minkowski sum $U \oplus V$ is defined by $U \oplus V = \{u+v : u \in U, v \in V\}$; the set $U\setminus V$ is defined by $U\setminus V = \{u : u \in U \, and \, u \notin V\}$; and the set $\lambda U = \{\lambda u : u \in U\}$ is a scaled set of $U$. Besides, int($U$) and $\partial U$ denote the interior and the boundary of $U$, respectively. Given a continuous function $\bar{\Psi} : \mathbb{R}^n \to \mathbb{R}$, and $\gamma \geq 0$, the level set $N[\bar{\Psi}, \gamma]$ is defined by $N[\Psi, \gamma] = \{x : \Psi(x) \leq \gamma\}$. The boundary of this set, $\partial N$, is the level surface of $\psi$.

## 2. System description

Let us consider the (controllable) system

$$\begin{aligned} x(k+1) &= Ax(k) + B\Delta u(k) \\ y(k) &= Cx(k), \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^{nx}$ is the current state vector, $\Delta u(k) = u(k) - u(k-1) \in \mathbb{R}^{nu}$ is the input move and $y(k)$ is the current output of the system. In addition, let us consider that the system has $nun$ unstable and $ns$ stables poles, and $ni = \max(nu, ny)$ poles at one. These integrating poles are produced by the incremental form of model (1). Then, matrix $A$ can be decomposed into the integrating, unstable and stable modes using the Jordan decomposition:

$$A = W\Lambda V = [\,W_i \quad W_u \quad W_s\,] \begin{bmatrix} \Lambda_i & 0 & 0 \\ 0 & \Lambda_u & 0 \\ 0 & 0 & \Lambda_s \end{bmatrix} \begin{bmatrix} V_i^T \\ V_u^T \\ V_s^T \end{bmatrix}, \tag{2}$$

where $WV = I_{nx}$, and $\Lambda$ is a block diagonal matrix (Jordan canonical form). The column spaces of matrices $W_i^T$, $W_u^T$ and $W_s^T$ are the integrating, unstable and stable subspaces or manifolds of the state space, $W_i$, $W_u$ and $W_s$, respectively. Since $W_i \oplus W_u \oplus W_s = \mathbb{R}^{nx}$, every state can be decomposed as $x = x^i + x^u + x^s$, where $x^i = W_iV_i^Tx$ belongs to $W_i$, $x^u = W_uV_u^Tx$ belongs to $W_u$ and $x^s = W_sV_s^Tx$ belongs to $W_s$. Clearly, if $x \in W_i$, then $x^u = W_uV_u^Tx = 0$ and $x^s = W_sV_s^Tx = 0$, and so on for the other subspaces. It can be shown that $W_i$, $W_u$ and $W_s$ are invariant subspaces of the state space under the transformation $A$. Although the results presented here can be extended to any linear system, we assume for simplicity that $\Lambda_i = I_{ni}$ (i.e., systems with non-repeated poles at one[1]).

The input is constrained to belong to a set $\Delta U$, defined as follows:

$$\begin{aligned} \Delta U = \{&\Delta u : -\Delta u_{\max} \leq \Delta u \leq \Delta u_{\max} \text{ and } u_{\min} \leq u(k-1) \\ &+ \Delta u \leq u_{\max}\}, \end{aligned}$$

where $u(k-1)$ is the past value of the input $u$. Furthermore, the states are constrained to be in a set $X$, which for simplicity is given by $X = X_i \oplus X_u \oplus X_s$, where $X_i \subseteq W_i$, $X_u \subseteq W_u$ and $X_s \subseteq W_s$. This set should be consistent with the operating window of the process.

**Remark 1.** Notice that in the model defined in (1), the input appears in the incremental form. This property gives an alternative way to achieve an offset-free control, in comparison to the target calculation strategy used in [15,16], that needs a separate optimization problem to be solved in order to compute the achievable steady-state. Secondly, input increment constraints are included together with the input constraints. This constitutes an important feature of real processes usually disregarded by theoretical MPC formulations, and contributes to a better description of the whole system, since this kind of constraints limits the maximal domain of attraction of the system, as can be seen in the next sub-section.

**Remark 2.** The evolution of input $u$ can be written as $u(k+1) = u(k) + \Delta u(k)$ and, from the decomposition (2), it follows that $V_i^Tx(k+1) = V_i^Tx(k) + V_i^TB\Delta u(k)$. Then $V_i^Tx = V_i^TBu$ and so $W_iV_i^Tx = x^i = W_iV_i^TBu$. Now, since the input amplitude is constrained, the state component $X^i$ should be constrained to belong to $X_i = \{x^i \in W_i : W_iV_i^TBu_{\min} \leq x^i \leq W_iV_i^TBu_{\max}\}$.

### 2.1. Characterization of the steady states

The idea here is to show that, under mild conditions, systems with integrating modes have the equilibrium states in the integrating manifold of the state space. Without loss of generality, it is assumed that $nu = ny = ni$ (for non-square systems the procedure to characterize the steady state is similar). For a given output target (or set-point) $y^{sp}$, any steady state of the system defined in (1), ($x_{ss}$, $\Delta u_{ss}$), associated with this target should satisfy $x_{ss} = Ax_{ss} + B\Delta u_{ss}$ and $y^{sp} = Cx_{ss}$. Then, using (2), pre-multiplying the first equation above by $V$, and considering that $WV = I_{nx}$, it follows that

$$\begin{bmatrix} (\Lambda - I_{nx}) & VB \\ CW & 0_{ni,ni} \end{bmatrix} \begin{bmatrix} Vx_{ss} \\ \Delta u_{ss} \end{bmatrix} = \begin{bmatrix} 0 \\ y^{sp} \end{bmatrix}.$$

Now, developing the matrices of the above equation into its integrating, unstable and stable components, we have

$$\begin{bmatrix} I_{ni} - I_{ni} & 0 & 0 & V_i^TB \\ 0 & \Lambda_u - I_{nun} & 0 & V_u^TB \\ 0 & 0 & \Lambda_s - I_{ns} & V_s^TB \\ CW_i & CW_u & CW_s & 0_{ni,ni} \end{bmatrix} \begin{bmatrix} V_i^Tx_{ss} \\ V_u^Tx_{ss} \\ V_s^Tx_{ss} \\ \Delta u_{ss} \end{bmatrix} = \begin{bmatrix} 0 \\ y^{sp} \end{bmatrix}. \tag{3}$$

Thus, assuming that $rank(V_i^TB) = ni$, it follows that any steady state of the system is such that

$$\begin{bmatrix} CW_iV_i^Tx_{ss} \\ V_u^Tx_{ss} \\ V_s^Tx_{ss} \\ \Delta u_{ss} \end{bmatrix} = \begin{bmatrix} y^{sp} \\ 0_{nun,1} \\ 0_{ns,1} \\ 0_{ni,1} \end{bmatrix}, \tag{4}$$

From Eq. (4) it can be seen that the unstable and stable components of $x_{ss}$ are null ($x_{ss}^u = W_uV_u^Tx_{ss} = 0$, $x_{ss}^s = W_sV_s^Tx_{ss} = 0$) and $\Delta u_{ss} = 0$. This means that the steady states are condensed in the subspace corresponding to the integrating modes, $W_i$ ($x_{ss}^i = W_iV_i^Tx_{ss} = x_{ss}$); and furthermore, $x_{ss}^i = W_i(CW_i)^{-1}y^{sp}$. Notice that the equilibrium states are confined to a $ni$ dimensional subspace, which is the output dimension; so, every point of the integrating manifold is an equilibrium state of the unconstrained system.

Now, since the system is subject to constraints, it should be steered to those steady states that satisfy the constraints. From the definition of the state constraints, the set of these admissible steady states is then given by $X_{ss} \equiv X_i \subseteq W_i$.

### 2.2. Characterization of the controllable sets

In this section we exploit the steady state characterization presented in the last section in order to define some useful controllable sets. First, we define the set of states in $X$ that can be admissibly steered, by means of an admissibly sequence of $j$ control actions, to

[1] Notice that when the original system has integrating poles, the velocity form of the model defined in (1) produces integrating modes with multiplicity two. In this case, matrix $\Lambda_i$ will be triangular, and the original integrating modes are considered as unstable modes (See [14]).

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the equilibrium set, $X_{ss}$ ($j$-step controllable set to $X_{ss}$):

$$C_j(X, X_{ss}) = \{x(0) \in C : \text{for all } k = 0, \ldots, j-1, \\ \exists \Delta u(k) \in \Delta U \text{ such that } x(k) \in X \text{ and } x(j) \in X_{ss}\}.$$

Since the target set is an equilibrium set, then $C_j$ is a control invariant set for the system (1) subject to $\Delta u \in \Delta U$ and $x \in X$, for all $j \geq 1$ (and so they are called stabilizable sets). The set[2] $C_\infty \subset X$ is therefore *the largest possible domain of attraction* of any controller designed for system (1), since it does not depend on the selected control law, but on the nature of the system and the constraints.

Now we will define the $j$-step controllable set to the equilibrium set $X_{ss}$, for unstable modes:

$$C_j^u(X_u, X_{ss}) = \{x^u(0) \in X_u : \text{for all } k = 0, \ldots, j-1, \\ \exists \Delta u(k) \in \Delta U \text{ such that } x(k) \in U,\ x(j) \in U \text{ and } x^u(j) \in X_{ss}\}$$

Clearly, the terminal condition $x^u(j) \in X_{ss}$ is equivalent to $x^u(j) \in \{0\}$, and so, $C_j^u$ defines the unstable state components that can be steered to the origin in $j$ steps, by means of a feasible control sequence $\{\Delta u(0), \ldots, \Delta u(j-1)\}$ and maintaining the state sequence $\{x(0), \ldots, x(j)\}$ in $X$. Notice that the projection of $C_j$ onto $W_u$ is a subset of $C_j^u$, since the latter set only forces the terminal components $x^u(j)$ to be null, while setting the terminal components $x^i(j)$ and $x^s(j)$ free inside $X_i$ and $X_s$, respectively. It can be shown that $C_j^u \subset C_{j+1}^u$, for all $j > 0$, and furthermore $C_j^u$ tends to a bounded set as the number of steps $j$ tends to infinity, even if $X$ is unbounded.[3]

**Remark 3.** The set $C_j^u$ can be computed using iteratively the well-known methods for computing the one step controllable set to a given set (quantifier elimination, projection and Minkowski summation, etc.) and the projection of a set onto a subspace (Fourier–Motzkin elimination, etc.) – see chapters 2 and 3 of [17] for details. Since the system is linear and the constraints are box-type constraints, the sets $C_j^u$ can be easily computed using available tools: Matlab Invariant set Toolbox [18]; and Multi-parametric Toolbox (MPT) [19]. From a practical point of view, the set $C_\infty^u$ can be computed by increasing the index $j$ of $C_j^u$ up to $N$, in such a way that $C_{N+1}^u \approx C_N^u$. Then, $C_N^u \approx C_\infty^u$, and this will be a good estimation of the largest possible domain of attraction for the unstable subspace.

Now, from the latter definition, one can define the following subset of $C_\infty \subset \mathbb{R}^{nx}$:

$$\Theta_j = \{x \in C_\infty : x^u \in C_j^u(X_u, X_{ss})\}.$$

This set satisfies $C_j \subseteq \Theta_j$ (in fact, $\Theta_j$ is significantly less conservative than $C_j$ for most of the cases, given that the projection of $C_j$ onto $W_u$ is a subset of $C_j^u$). Furthermore, it can be shown that $\Theta_j$ is a control invariant set for system (1) subject to $\Delta u \in \Delta U$ and $x \in X$.

### 2.3. Controllable sets in the context of MPC

It is well-known from the MPC literature that when an infinite output horizon is used in the objective function of the controller, and no terminal controller is assumed for predictions beyond the control horizon, the non-stable (integrating and unstable) modes must be canceled at the end of the control horizon, $m$, to prevent unbounded values of the cost. Then, unless the current state is such that its integrating-unstable component is in the $m$-controllable set to the equilibrium point, the infinite horizon MPC will be infeasible. For the case of integrating modes, the required terminal constraint can be easily slacked and then, by means of a constraint violation penalty, the asymptotic convergence of these modes to zero is guaranteed, together with the convergence of the state to the equilibrium point [20]. On the other hand, the terminal constraint associated to the pure unstable modes cannot be directly slacked, since for this case a constraint violation penalty does not produce an asymptotic convergence of these modes to zero. In this context, it is useful to characterize the maximal region where the terminal constraint for the unstable modes is feasible. From the analysis of controllable sets in Section 2.2, it follows that this set is given by $\Theta_m$.

## 3. A stable MPC for tracking

### 3.1. Conventional infinite horizon MPC (IHMPC)

For tracking a non-zero output target $y^{sp}$, the classical IHMPC formulation is as follows [7,16]:

Problem 1

$$\min_{\Delta u_k} J_k = \sum_{j=0}^{m-1} \{\|x(k+j|k) - x^{sp}\|_Q^2 + \|\Delta u(k+j|k)\|_R^2\} + \|x(k+m|k)\|_P^2$$

subject to :

$$\Delta u(k+j|k) \in \Delta U, \quad j = 0, \ldots, m-1$$

$$x(k+j|k) \in X, \quad j = 1, \ldots, m$$

$$x^i(k+m|k) - x^{sp} = 0 \tag{5}$$

$$x^u(k+m|k) = 0 \tag{6}$$

where $x(k|k) = x(k)$, $x^{sp} = W_i(CW_i)^{-1}y^{sp}$[4], $\Delta u(k+j|k)$ is the control move computed at time $k$ to be applied at time $k+j$, $m$ is the control horizon, $Q$ and $R$ are positive weighting matrices of appropriate dimension, and $\Delta u_k = (\Delta u(k|k), \ldots, \Delta u(k+m-1|k))$. Observe that $x^{sp}$ is the equilibrium state corresponding to the output set point $y^{sp}$. Because of the terminal constraints (5) and (6), which correspond to the zeroing of the integrating and unstable modes at the end of the control horizon respectively, it can be shown that the control cost of Problem 1 corresponds to the infinite horizon cost if the terminal penalty matrix $P$, that depends only on the stable modes of system, is computed by solving a Lyapunov equation as in [11].

The domain of attraction of the controller resulting from the solution to Problem 1 may be small (see [16] for an interesting example), mainly in the cases where the input increment constraint is included. As discussed in Section 2.3, the inclusion of slack variables in the terminal constraints is appropriate for constraint (5), which is related to the integrating modes (see [20,14]). However, it is not appropriate to remove the conflict between constraint (6) and the input constraints to preserve the stability of the system, and so, a new method has to be developed to maximize the domain of attraction of the infinite horizon MPC.

### 3.2. The proposed stable MPC with maximal domain of attraction

To enlarge the domain of attraction of the controller defined through Problem 1, in Ref. [11], it is presented a method to include

[2] In the sequel, the set $C_j(X, X_{ss})$ will be simply denoted as $C_j$.

[3] As it was done with the unstable modes, it could be defined the $j$-step controllable set to the equilibrium set for the integrating-unstable modes and for stable modes as $C_j^{i-u}(X_{i-u}, X_{ss})$ and $C_j^s(X_s, X_{ss})$, respectively ($X_{i-u}$ is the direct sum of $X_i$ and $X_u$). It can be shown that $C_j^{i-u}$ always tends to a bounded set as $j$ tends to infinity (see Remark 1 of [11]). On the other hand, if $C_s \equiv W_s$, the set $C_\infty^s$ is necessarily unbounded.

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slack variables in constraints (5) and (6), while preserving the stability of the closed-loop system. The main idea of this approach is to separate the IHMPC problem in two stages: the first one is devoted to steer the state to the set $\Theta_m$ (maximal region where the terminal constraint for unstable modes is feasible) in a finite number of steps; while the second one is devoted to asymptotically steer the state from $\Theta_m$ to the desired equilibrium point in $X_{ss}$, as classical IHMPC does. Each of these stages corresponds to a different optimization problem and there is guarantee of feasibility and boundness of the cost function at any time step. However, the strategy presented in Ref. [11] has some limitations, since it cannot be applied to systems with an arbitrary number of unstable modes.

The idea here is to generalize the two-stage procedure presented in Ref. [11] to the case where we may have an arbitrary number of unstable modes. For this purpose, it should be emphasized that the objective of the first stage of the two-stage IHMPC formulation is to steer the state from $\Theta_j$ to $\Theta_{j-1}$, for $m<j\leq N$ and $N$ computed as in Remark 4, until $\Theta_m$ is reached. So, it makes sense to use, as the cost function of the first optimization problem, a function of $x^u(k+m|k)$ whose level surfaces matches in some way the contours of the controllable sets $\{C^u_m, \ldots, C^u_N\}$. With this approach, to minimize the latter cost becomes equivalent to feasibly steer the states towards $\Theta_m$. A $l_2$-norm of $x^u(k+m|k)$ would not produce the desired effect (except for the case where each input is associated with at most one unstable mode, as it was assumed in Ref. [11]), since the level surfaces of this norm are different in shape from the contours of the controllable sets. Here, we propose the development of a new functional, which is derived from the Minkowski functional. It will be shown that the proposed functional directly associates their level surfaces with the contour of scaled sets of $C^u_j$. As a consequence, a new IHMPC with maximal domain of attraction is obtained.

### 3.2.1. The generalized Minkowski functional

This subsection is devoted to the introduction of a new functional (that we call *generalized Minkowski functional*) which is an extension of the well-known Minkowski functional. The Minkowski functional is defined as follows [21]:

**Definition 1.** Given a convex set $S\subset X$ that includes the origin as an interior point, the Minkowski functional $\Psi_S$ associated to $S$ is defined as

$$\Psi_S(x)=\inf\{\mu\geq 0 : x\in\mu S\}. \tag{7}$$

The main property of this (convex) functional is that its level sets have the shape of the set $S$, i.e., the level sets of $\Psi_S(x)$ can be obtained by linearly scaling the set $S$. Observe that from the definition (7), $S=N[\Psi_S(x), 1]$, $\Psi_S(x)<1$, for $x\in\operatorname{int}(S)$, and $\Psi_S(x)>1$, for $x\notin S$. Fig. 1 shows the set $S$ and the level sets of $\Psi_S(x)$, for an example in $\mathbb{R}^2$.

The idea here is to extend the concept of the Minkowski functional associated to a set, to a new generalized functional associated to a sequence of sets. For this purpose, let us consider a sequence of 0-symmetric convex sets $\{S_1, \ldots, S_n\}$, with $S_1\subset S_2\subset\ldots\subset S_n$, and consider the corresponding sequence of pairwise disjoint sets, $\{\Upsilon_1, \ldots, \Upsilon_n\}$, where $\Upsilon_{i+1}=\operatorname{int}(S_{i+1})\setminus\operatorname{int}(S_i)$, $i=1, \ldots, n-1$ (observe that these sets are a partition of $S_n$). Then, the aim is to find a functional whose level surfaces in $\Upsilon_{i+1}$ are some scaled sets of $S_i$, for $i=1, \ldots, n-1$. Fig. 2 shows the sequence of sets $\{S_1, S_2, S_3\}$, the corresponding sequence $\{\Upsilon_2, \Upsilon_3\}$ and the level surfaces that the desired functional should have, for an example in $\mathbb{R}^2$. In this context, an intuitive first candidate for the desired generalized functional could be as follows:

Given a sequence of convex sets $\{S_1, \ldots, S_n\}$, with $S_1\subset S_2\subset\ldots\subset S_n$, the generalized Minkowski functional $\Psi'_{\{S_1,\ldots,S_n\}}$ associated to $\{S_1, \ldots, S_n\}$ is defined as

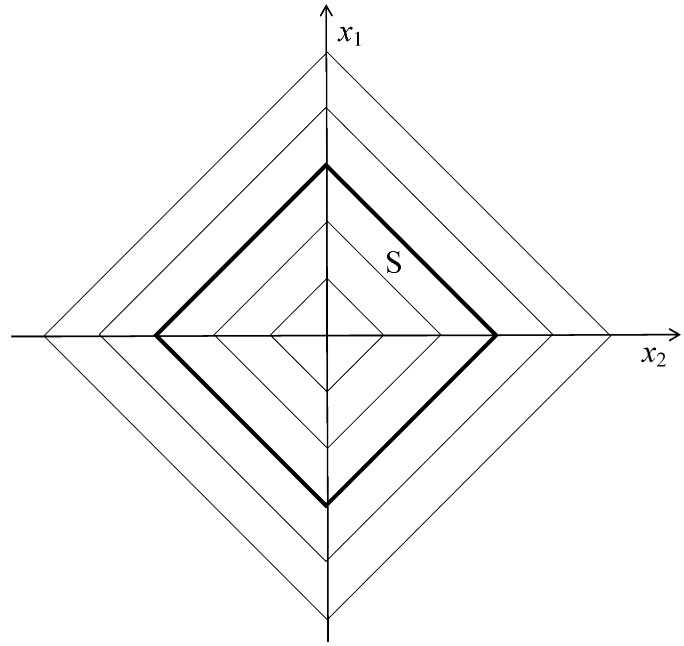


**Fig. 1.** The level sets of the Minkowski functional $\Psi_S(x)$.

$$\Psi'_{\{S_1,\ldots,S_n\}}(x)=\begin{cases}\Psi_{S_i}(x) & \text{if } x\in\Upsilon_{i+1},\quad i=1,\ldots,n-1\\ \Psi_{S_1}(x) & \text{if } x\in S_1\end{cases}$$

This functional, however, does not have the desired property; i.e., its level sets are not the scaled sets of $S_i$. It looks like the one shown in the simplified diagram of Fig. 3. Furthermore, it is not a quasi-convex function and so it cannot be directly used as the cost function of an optimization problem. In fact, given that $S_1\subset S_2\subset\ldots\subset S_n$, the sequence of functionals $\{\Psi_{S_1}(x), \ldots, \Psi_{S_n}(x)\}$ is such that $\Psi_{S_1}(x)\geq\Psi_{S_2}(x)\geq\ldots\geq\Psi_{S_n}(x)$ for each $x$.

To overcome this difficulty, it is possible to replace the sequence of Minkowski functionals $\{\Psi_{S_1}(x), \ldots, \Psi_{S_n}(x)\}$ by

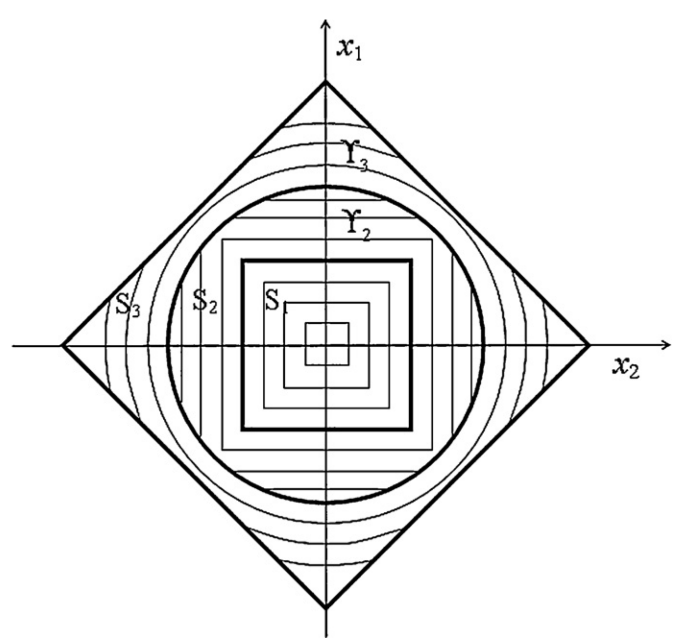


**Fig. 2.** Sequence of sets $\{S_1, S_2, S_3\}$ and level surfaces of the desired generalized functional.

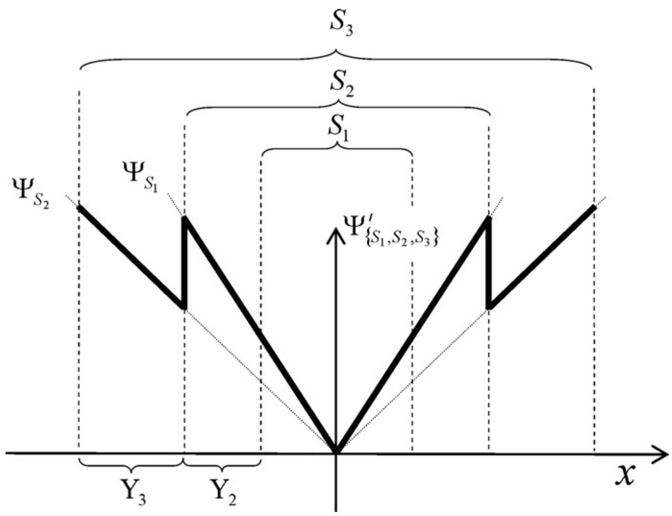


**Fig. 3.** Descriptive diagram of the generalized Minkowski functional $\Psi'_{\{S_1,S_2,S_3\}}(x)$.

$\{\Psi_{\nu_1 S_1}(x), \ldots, \Psi_{\nu_n S_n}(x)\}$, where the sequence of real numbers $\{\nu_1, \ldots, \nu_n\}$ is such that $\nu_1 S_1 \supseteq \nu_2 S_2 \supseteq \ldots \supseteq \nu_n S_n$ (see Fig. 4). In this way, we have $\Psi_{\nu_1 S_1}(x) \leq \Psi_{\nu_2 S_2}(x) \leq \ldots \leq \Psi_{\nu_n S_n}(x)$, for each $x$, and the following generalized functional can be defined:

**Definition 2.** Given a sequence of convex sets $\{S_1, \ldots, S_n\}$, with $S_1 \subset S_2 \subset \ldots \subset S_n$, we propose to define the generalized Minkowski functional $\Psi_{\{S_1,\ldots,S_n\}}$ associated to $\{S_1, \ldots, S_n\}$ as follows

$$\Psi_{\{S_1,\ldots,S_n\}}(x) = \begin{cases} \Psi_{\nu_i S_i}(x) & \text{if} \quad x \in \Upsilon_{i+1}, \quad i = 1, \ldots, n-1 \\ \Psi_{\nu_1 S_1}(x) & \text{if} \quad x \in S_1 \end{cases}$$

where the coefficients $\nu_i$ are such that $\nu_i S_i \supseteq \nu_{i+1} S_{i+1}$.

This generalized Minkowski functional looks like the one shown in the simplified diagram of Fig. 5, which is a quasi-convex function. Furthermore, its level surfaces in each set $\Upsilon_{i+1}$ are scaled sets of $S_i$, for $i = 1, \ldots, n-1$, as desired (see Fig. 2). More technically, the proposed generalized Minkowski functional satisfies the following properties:

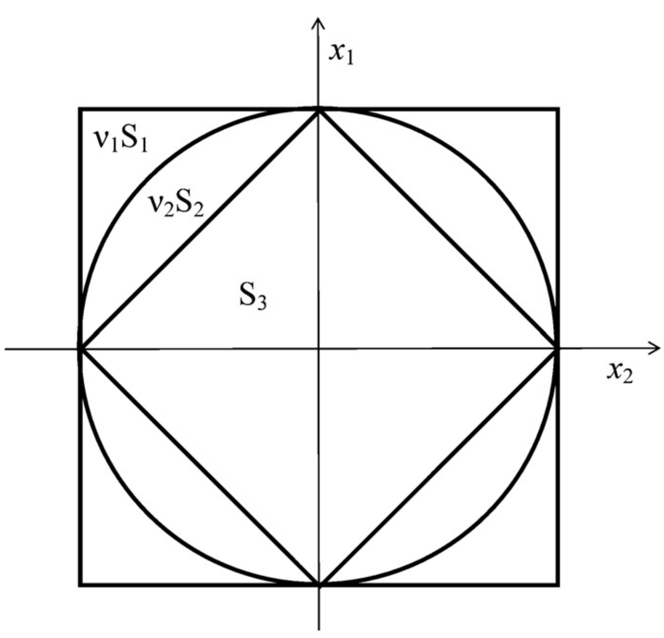


**Fig. 4.** Sequence of scaled sets $\{\nu_1 S_1, \nu_2 S_2, \nu_3 S_3\}$, corresponding to the sequence $\{S_1, S_2, S_3\}$ of Fig. 2, for $\nu_1 > \nu_2 > \nu_3$ and $\nu_3 = 1$.

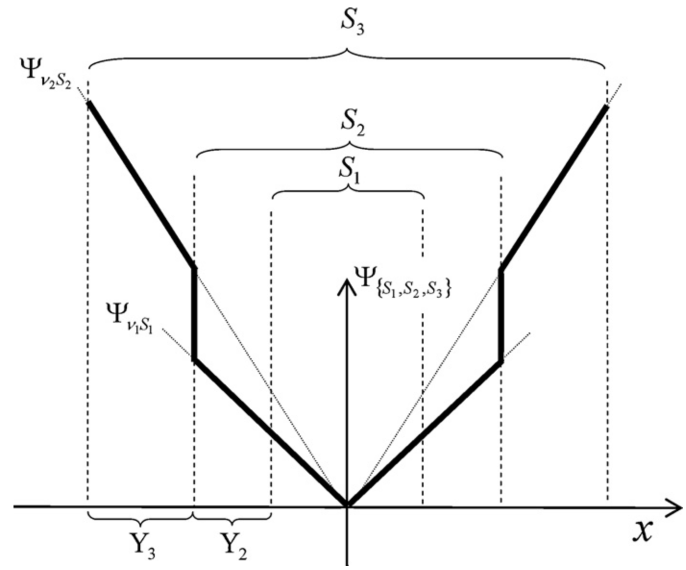


**Fig. 5.** Descriptive diagram of the quasi-convex generalized Minkowski functional $\Psi_{\{S_1,S_2,S_3\}}(x)$.

Properties:

1. $\Psi_{\{S_1,\ldots,S_n\}}(x)$ is piecewise continuous and piecewise convex. It has a discontinuity at each contour $\partial S_i$ of the sets $S_i$, for $i = 1, \ldots, n-1$. Furthermore, $\Psi_{\{S_1,\ldots,S_n\}}(x) > 0, \forall x \neq 0$, and $\Psi_{\{S_1,\ldots,S_n\}}(x) = 0 \Leftrightarrow x = 0$.
2. $\Psi_{\{S_1,\ldots,S_n\}}(x)$ has a global minimum at 0. This follows from the fact that every Minkowski functional $\Psi_{\nu_i S_i}(x)$ has a global minimum at 0.
3. If $x \in \Upsilon_i$, then $\Psi_{\{S_1,\ldots,S_n\}}(x) = 1/\nu_{i-1} \Leftrightarrow x \in \partial S_{i-1}$ for $i = 2, \ldots, n$. That is: $\partial N[\Psi_{\{S_1,\ldots,S_n\}}(x), 1/\nu_i] = \partial S_i$, for $i = 1, \ldots, n-1$, and as a consequence, on the contours of the sets $S_i$, the generalized Minkowski functional has increasing values.
4. $\Psi_{\{S_1,\ldots,S_n\}}(x)$ satisfies:

   $$\Psi_{\{S_1,\ldots,S_n\}}(x_1) \leq \Psi_{\{S_1,\ldots,S_n\}}(x_2) \leq \Psi_{\{S_1,\ldots,S_n\}}(x_3)$$

   for $x_1 \in \Upsilon_i$, $x_2 \in \partial S_i$ and $x_3 \in \Upsilon_{i+1}$ which means that $\Psi_{\{S_1,\ldots,S_n\}}(x)$ is strictly increasing according to the following definition:

   $$N[\Psi_{\{S_1,\ldots,S_n\}}(x), \gamma_1] \subset N[\Psi_{\{S_1,\ldots,S_n\}}(x), \gamma_2] \Leftrightarrow \gamma_1 < \gamma_2.$$

5. If all the sets $S_i$ have the same aspect ratio (that is, if $\lambda_i S_i = S_n$, $i = 1, \ldots, n$ for some sequence of coefficients $\lambda_i$) then, the generalized Minkowski functional becomes the Minkowski functional corresponding to the largest set, $S_n$.
6. *Strictly quasi-convexity*: The proposed functional is not convex. However, it can be shown that $\Psi_{\{S_1,\ldots,S_n\}}(x)$ is strictly quasi-convex, that is $\Psi_{\{S_1,\ldots,S_n\}}(\lambda x_1 + (1-\lambda)x_2) < \max(\Psi_{\{S_1,\ldots,S_n\}}(x_1), \Psi_{\{S_1,\ldots,S_n\}}(x_2))$, for $\lambda \in (0, 1)$ and $x_1 \neq x_2$, $\Psi_{\{S_1,\ldots,S_n\}}(x_1) \neq \Psi_{\{S_1,\ldots,S_n\}}(x_2)$.

The proofs of the these properties were omitted for brevity.

### 3.2.2. The proposed MPC formulation

Now, the generalized Minkowski functional presented above can be used as the objective function of the first problem of the two step IHMPC, where the objective is to feasibly steer the unstable component of the state to the set $C^u_m$. Then, the proposed infinite horizon MPC is obtained from the sequential solution of the following two problems:

Problem 2a

$$\min_{\Delta u_{a,k}} J_{a,k} = \Psi_{\{C_m^u,\ldots,C_N^u\}}(x^u(k+m|k))$$

Subject to:

$$\Delta u_a(k+j|k) \in \Delta U, \quad j=0,\ldots,m-1$$

$$x(k+j|k) \in X, \quad j=1,\ldots,m$$

Problem 2b

$$\min_{\Delta u_{b,k},\delta_k^i,\delta_k^u} J_{b,k} = \sum_{j=0}^{m-1} \{||x(k+j|k) - x^{sp} - \delta(k,j)||_Q^2$$

$$+||\Delta u_b(k+j|k)||_R^2\} + ||x(k+m|k)_P^2 + J^i(\delta_k^i)$$

Subject to:

$$\Delta u_b(k+j|k) \in \Delta U, \quad j=0,\ldots,m-1$$

$$x(k+j|k) \in X, \quad j=1,\ldots,m$$

$$x^i(k+m|k) - x^{sp} = \delta_k^i \tag{8}$$

$$x^u(k+m|k) = \delta_k^u \tag{9}$$

$$\Psi_{\{C_m^u,\ldots,C_N^u\}}(x^u(k+m|k)) \le \Psi^* \tag{10}$$

where $\delta_k^i$ and $\delta_k^u$ defined in (8) and (9) are slack variables associated with the integrating and unstable modes, respectively; the constraint violation penalty, $J^i(\delta_k^i) : \mathbb{R}^{ni} \to \mathbb{R}$, is a convex, positive definite function such that $J^i(0)=0$; and $\Psi^*$ is the optimal cost of Problem 2a. Because of the inclusion of a constraint violation term $\delta(k,j) = \delta_k^i + W_u \Lambda_u^{j-m} V_u^T \delta_k^u$ into the predictions, the infinite horizon cost of Problem 2b can be also written as a finite horizon cost with a terminal penalty term (see [10,11] for the details). Constraint (10) forces the states to remain in the controllable set determined by Problem 2a. Clearly, as $\Psi^*$ decreases, the slack variable $\delta_k^u$ tends to zero.

The following algorithm produces a stabilizing control law:

**Algorithm 1.** At every time step, $k$, solve Problem 2a and pass the optimal value of the cost to Problem 2b. Implement the first control action $\Delta u_b(k|k)$.

**Remark 4.** From a practical point of view, the computation of the generalized Minkowski functional, $\Psi_{\{C_m^u,\ldots,C_N^u\}}(x^u)$, is as follows. First, the sequence of sets $\{C_m^u, \ldots, C_N^u\}$ is computed as in Remark 3. Then the sequence of coefficients $\{\nu_m, \ldots, \nu_N\}$ is computed by solving the optimization problems: $\nu_i = \min\{\nu \ge 0 : \nu C_i^u \supseteq \nu_{i+1} C_{i+1}^u\}$, for $i=m, \ldots, N-1$, with $\nu_N = 1$. It is important to note that the sets $\{C_m^u, \ldots, C_N^u\}$ and the coefficients $\{\nu_m, \ldots, \nu_N\}$ are computed off line, since they depend only on the system and the constraints, i.e., they are not controller parameters. Finally, following Definition 2, an algorithm is developed to evaluate this functional and use it as a cost function of Problem 2a.

**Remark 5.** Let us define the $m$ controllability matrix of system (1) as $Co_m = [\,A^{m-1}B \quad A^{m-2}B \quad \ldots \quad B\,]$. Then, once the state is in $\Theta_m$, the terminal constraints (8) and (9) in Problem 2b can be written as

$$\begin{bmatrix} Co_m^i \\ Co_m^u \end{bmatrix} \Delta u_{b,k} = \begin{bmatrix} \delta_k^i + x^{sp} - W_i V_i^T x(k) \\ -W_u V_u^T A^m x(k) \end{bmatrix},$$

where $Co_m^i = W_i V_i^T Co_m$ ($m$ controllability matrix for the integrating modes), $Co_m^u = W_u V_u^T Co_m$ ($m$ controllability matrix for the unstable modes). As a result, the control horizon $m$ should be large enough to assure that matrix $\begin{bmatrix} Co_m^i \\ Co_m^u \end{bmatrix}$ is full rank.

**Remark 6.** As optimization problems 2a and 2b are solved sequentially at every time step, we can assure that, depending on the available degrees of freedom of the system, while the state is being driven to $\Theta_m$, the transient performance will also be optimized in Problem 2b.

**Remark 7.** There exist different alternatives to define the slack penalization function $J^i(\delta_k^i)$. In Ref. [9] it is shown that $J^i$ must be positive definite and subdifferentiable to minimize the distance between the steady state output and the output set point. Two possible choices for $J^i$ that fulfill these conditions are the $l_1$-norm (or $l_\infty$-norm) penalty [9]: $J^i(\delta_k^i) = \alpha||\delta_k^i||_1$, where $\alpha$ is a real penalization; and the quadratic penalty [11]: $J^i(\delta_k^i) = ||\delta_k^i||_S^2$, where $S$ is a positive penalization matrix. In the first case, if a lower bound for $\alpha$ is found, the dynamic optimality of the solution is preserved (i.e., the slacked problem gives exactly the same solution as the original problem, if the latter has a feasible solution). However, a lower bound for the parameter $\alpha$ is difficult to obtain since it depends on the Lagrangian multipliers and so, it depends on the current states and output set-points [22]. In the second case, which is the choice in the present work, no exact penalty is achieved. The resulting lack of optimality, however, is not so relevant in the context considered here, given that the main objective here is to enlarge the domain of attraction of a controller that can be easily implemented in practice.

The theorem that follows proves the asymptotic stability of the controlled system, when the control law is obtained through the solution of problems 2a and 2b, and a quadratic slack penalization is used in Problem 2b.

**Theorem 1.** For system (1) subject to $\Delta u \in \Delta U$ and $x \in X$, the controller resulting from the application of Algorithm 1, with $J^i(\delta_k^i) = ||\delta_k^i||_S^2$, is always feasible with a domain of attraction given by $C_\infty$. Also, the control sequence obtained by applying Algorithm 1 at successive time steps drives the output of the closed loop system asymptotically to a point that minimizes $J^i$ (particularly, if the output set-point is reachable, the output of the closed loop system is asymptotically steered to the set-point without offset).

**Proof** *(:)*. The proof is divided into three parts: first, it will be shown that the state (admissibly) reaches the set $\Theta_m$ in a finite number of time steps; next it will be shown that once the state is in $\Theta_m$, the system is steered to a steady state given by $x_{ss} = x^{sp} + \delta_k^{i*}$, and finally it will be shown that the latter steady state minimizes $J^i(\delta_k^i)$ (with $\delta_k^{i*} = 0$ if the output set-point is reachable).

Part I

Let us assume that the output reference is reachable and the current state is such that $x \in C_\infty$. Consider first that at time step $k$, $x \in \Theta_m$, which implies that the unstable component of $x$, $x^u$, belongs to $C_m^u$. So, because of the definition of $C_j^u$, there exists a feasible control sequence that steers (admissibly) $x^u$ to zero in $m$ steps. Therefore, the optimal cost of Problem 2a, $J_{a,k}^* = \Psi^*$, will be null (see the property "(1)" of the generalized Minkowski functional). Once the null value of the cost of the first problem is passed to Problem 2b, constraint (10) forces the predicted state $x(k+m|k)$ to be in the integrating-stable subspace. Then, the sequence of Problems 2a and 2b is equivalent to Problem 2b with a feasible terminal constraint for the unstable modes (i.e., $\delta_k^u = 0$).

If at time step $k$, $x \notin \Theta_m$, then the optimal cost of Problem 2a will be $J_{a,k}^* = \Psi^* > 0$. This is so because by definition of the controllable sets for the unstable modes there is not a feasible control sequence that steers the unstable component of the state to zero in $m$ steps. Let us assume that the state $x(k|k)$

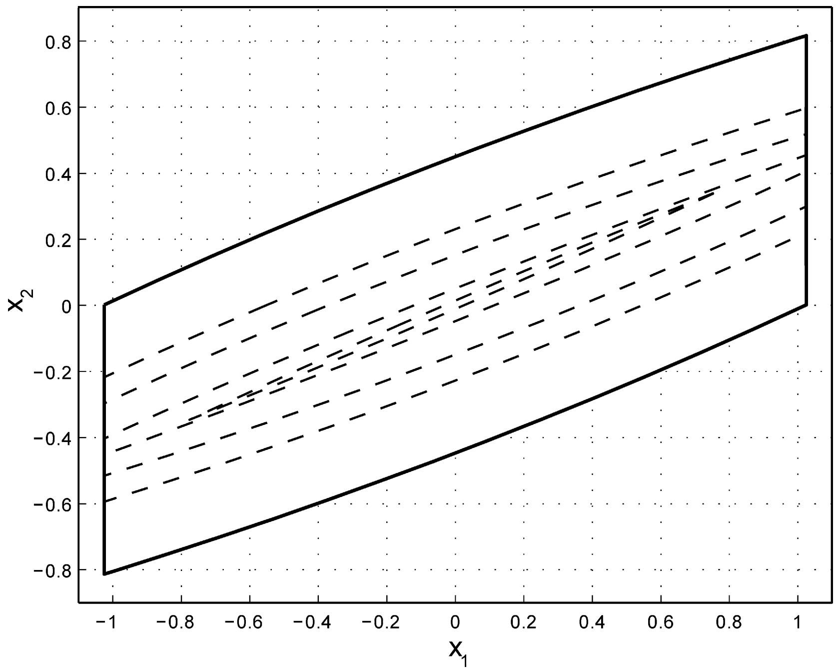


**Fig. 6.** Domain of attraction of the proposed controller (solid-line) and the conventional IHMPC (dashed-line).

belongs to $\Theta_n \setminus \Theta_{n-1}$, for some $m < n \le N$, which implies that $x^u(k|k)$ belongs to $C^u_n \setminus C^u_{n-1}$. Then, there exists a feasible input sequence that steers (admissibly) the unstable component to $C^u_{n-m}$. Furthermore, the region $C^u_{n-m}$ corresponds to the lowest possible value of the cost $\Psi_{\{C^u_m,\ldots,C^u_N\}}(x^u(k+m|k))$, since the level surfaces of the cost function matches the contours of the controllable sets (see the property "(4)" of the generalized Minkowski functional) and $C^u_{n-m} \subset C^u_{n-m+1} \subset \cdots \subset C^u_n$. Then, the optimal input sequence at time $k$ will be such that $x^u(k+1|k) \in C^u_{n-1}$, $x^u(k+2|k) \in C^u_{n-2}, \ldots, x^u(k+m|k) \in C^u_{n-m}$. Once the optimal value of the cost of the first problem, $\Psi^*$, is passed to Problem 2b, constraint (10) forces the predicted unstable component $x^u(k+m|k)$ to stay in $C^u_{n-m}$. Furthermore, every predicted unstable component will belong to the controllable set specified by Problem 2a; this is so, because by the definition of controllable set, it is not possible to reach $C^u_{n-m}$ in $m$ steps, without reaching $C^u_{n-m+1}$ in $m-1$ steps, and so on until $C^u_{n-1}$ is reached in one step. Now, given that no model mismatch is considered, then $x^u(k+1|k) = x^u(k+1|k+1)$, which means that the actual unstable component $x^u(k+1|k+1)$ belongs to $C^u_{n-1}$. Following this reasoning, the unstable component of the state will reach (admissibly) $C^u_m$ in $n-m$ steps (i.e., $x^u(k+n-m|k+n-m) \in C^u_m$), which implies that $x \in \Theta_m$, and the optimal cost of Problem 2a, $\Psi^*$, goes to zero in $n-m$ time steps. Again, because of constraint (10), once $\Psi^*$ is zeroed, Problems 2a and 2b become equivalent to Problem 2b with a feasible terminal constraint for the unstable modes.

Part II

Now, we will show that if the state is in $\Theta_m$ (i.e., $\Psi^* = 0$ and $\delta^u_k = 0$), then the solution to Problem 2b produces a control sequence that steers the states to a steady state given by $x_{ss} = x^{sp} + \delta^{i*}_k$. As usual in MPC, one can compare successive optimal costs $J^*_{b,k}$ and $J^*_{b,k+1}$ by defining a feasible input sequence, $\Delta\tilde{u}_{k+1} = (\Delta u^*(k+1|k), \ldots, \Delta u^*(k+m-1|k), 0)$, and a feasible slack variable $\tilde{\delta}^i_{k+1} = \delta^{i*}_k$, in order to show that $J^*_{b,k}$ is a Lyapunov function. Following a similar procedure to that of [23], it can be shown that

$$\tilde{J}_{b,k+1} = J^*_{b,k} + \|x(k|k) - x^{sp} - \delta^{i*}_k\|^2_Q + \|\Delta u^*_b(k|k)\|^2_R,$$

where $\tilde{J}_{b,k+1}$ is the cost obtained with the feasible solution proposed above, the optimal cost at time $k+1$ will be such that

$$J^*_{b,k+1} \le J^*_{b,k} + \|x(k|k) - x^{sp} - \delta^{i*}_k\|^2_Q + \|\Delta u^*_b(k|k)\|^2_R,$$

This last equation means that at the steady state, the closed loop state and input will be such that $\|x(k|k) - x^{sp} - \delta^{i*}_k\|^2_Q = 0$ and $\|\Delta u^*_b(k|k)\|^2_R = 0$, which implies that $x(k|k) = x^{sp} + \delta^{i*}_k$.

Part III

To complete the proof, it will be shown that the slack variable penalization (i.e., the proposed quadratic penalization) is minimized at steady state. For this purpose, consider a steady state (time $\bar{k}$ large enough) given by $x(\bar{k}|\bar{k}) = x^{sp} + \delta^i_{\bar{k}}$ and $\Delta u_{b,\bar{k}} = (0, \ldots, 0)$. Since a steady state is considered, then $x(\bar{k}|\bar{k}) \in W_i$. The optimization cost corresponding to that point will be $J_{b,\bar{k}} = \|\delta^i_{\bar{k}}\|^2_S$, and given that the sequence of control actions are null, the terminal constraints corresponding to this point will (trivially) be

$$x(\bar{k}|\bar{k}) - x^{sp} = \delta^i_{\bar{k}} \tag{11}$$

$$W_u V_u^T A^m x(\bar{k}|\bar{k}) = 0. \tag{12}$$

Now, because of the rank condition discussed in Remark 6, there exists a sequence of $m$ small changes in the manipulated variable, $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_{m-1}) \in \mathbb{R}^{m \cdot nu}$, that moves the system in the direction of the set-point (i.e., produces a decrease of the integrating slack variable), maintaining the predicted unstable component $x^u(k+m|k)$ at zero. If we now implement this small changes at time $\bar{k}$ (instead of the null increment that corresponds to steady-state situation) the terminal constraints become

$$x(\bar{k}|\bar{k}) + Co^i_m \varepsilon - x^{sp} - \bar{\delta}^i_{\bar{k}} = 0, \tag{13}$$

$$W_u V_u^T A^m x(\bar{k}|\bar{k}) + Co^u_m \varepsilon = 0, \tag{14}$$

where the new slack variable $\bar{\delta}^i_k$ is such that $\|\bar{\delta}^i_{\bar{k}}\|^2_S < \|\delta^i_{\bar{k}}\|^2_S$. These small changes also produce an increase in the dynamic error. To analyze the effect of this increase in the cost function of Problem 2b, let us consider the predicted state errors:

$$x(\bar{k}+j|\bar{k}) - x^{sp} - \bar{\delta}^i_{\bar{k}} = A^j x(\bar{k}|\bar{k}) + [\,Co_j \quad O_j\,]\,\varepsilon - x^{sp} - = \bar{\delta}^i_{\bar{k}} \tag{15}$$

**Table 1**
Domain of attraction of typical MPC strategies.

| MPC strategy | Domain attraction | Dependence on controller parameter |
|---|---|---|
| IHMPC [7] | $\{x \in X : x^{i-u} \in C_m^{i-u}(X, \{x^{sp}\})\}$ | $m, x^{sp}$ |
| Slacked IHMPC [23] | $\Theta_m$ | $m$ |
| Dual MPC [1] | $C_m(X, O_\infty^K(x^{sp}))$ | $m, K, x^{sp}$ |
| Dual MPC for tracking [8] | $C_m(X, O_\infty^K)$ | $m, K$ |
| Proposed IHMPC | $C_\infty$ | – |

where $O_j = 0_{j.nu,(m-j).nu}$. Since $A^j x(\bar{k}|\bar{k}) \in W_i$, then $A^j x(\bar{k}|\bar{k}) = x(\bar{k}|\bar{k})$; and taking into account (13), the predicted state errors can be written as

$$x(\bar{k}+j|\bar{k}) - x^{sp} - \bar{\delta}_{\bar{k}}^i = ([\, Co_j \quad O_j \,] - Co_m^i)\varepsilon \tag{16}$$

which depends exclusively on $\varepsilon$. As a result, the cost function of Problem 2b can be expressed as a function of $\varepsilon$ as follows[4]:

$$\bar{J}_{b,\bar{k}}(\varepsilon) = ||\varepsilon||_{\tilde{Q}}^2 + ||\delta_{\bar{k}}^i + Co_m^i \varepsilon||_S^2,$$

where $\tilde{Q} = M_1^T \bar{Q} M_1 + \bar{R} + M_2^T P M_2$, $\bar{Q} = \mathrm{diag}(Q, \ldots, Q)$, $\bar{R} = \mathrm{diag}(R, \ldots, R)$ and

$$M_1 = \begin{bmatrix} [Co_0 \quad O_0] - Co_m^i \\ \vdots \\ [Co_m \quad O_m] - Co_m^i \end{bmatrix}, \quad M_2 = Co_{m+1} - Co_m^i$$

This is the cost obtained with the proposed small change, and trivially satisfies $\bar{J}_{b,\bar{k}}(\varepsilon = 0) = J_{b,\bar{k}} = ||\delta_{\bar{k}}^i||_S^2$. Now, to know if the new cost is smaller than the previous one ($J_{b,\bar{k}}$) for small values of $\varepsilon$, let us consider the derivative of the cost function with respect to the change:

$$\begin{aligned}\frac{\partial \bar{J}_{b,\bar{k}}(\varepsilon)}{\partial \varepsilon} &= \varepsilon^T(\tilde{Q}^T + \tilde{Q}) + (\delta_{\bar{k}}^i + Co_m^i \varepsilon)^T (S^T + S) Co_m^i \\ &= 2\varepsilon^T \tilde{Q} + 2(\delta_{\bar{k}}^i + Co_m^i \varepsilon)^T S Co_m^i,\end{aligned}$$

which represents a vector of $m \cdot nu$ components. Notice that the second term of the right hand side of the above equation is negative because $\varepsilon$ is such that $||\bar{\delta}_{\bar{k}}^i||_S^2 < ||\delta_{\bar{k}}^i||_S^2$, and so $\partial ||\bar{\delta}_{\bar{k}}^i(\varepsilon)||_S^2 / \partial \varepsilon < 0$. The derivative evaluated at $\varepsilon = 0$ gives

$$\left.\frac{\partial \bar{J}_{b,\bar{k}}(\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} = 2\delta_{\bar{k}}^{i^T} S Co_m^i < 0.$$

This means that there exists a (possibly small) value of $\varepsilon$ such that $\bar{J}_{b,\bar{k}} < J_{b,\bar{k}}$. Therefore, a decrement in the slack penalty corresponds to a decrement in the complete MPC cost, and so the controlled system asymptotically converges to a steady state output that minimizes the distance to the output set-point. As a consequence of the latter property, if the output set-point is reachable, then the MPC cost asymptotically converges to zero. □

**Remark 8.** The main advantage of the proposed controller is that it has the largest possible domain of attraction, i.e., the domain of attraction is given by $C_\infty$, which does not depend on the control law.

Table 1 below shows a comparison between the domain of attraction of the proposed IHMPC and other typical strategies.[5]

[4] The form of this term results from (13) and (11).

[5] The set $O_\infty^K(x^{sp})$ is the maximal admissible invariant set for a given control law $u = Kx$ (LQR) and a given equilibrium point $x^{sp}$. The set $O_\infty^K$ is the maximal admissible invariant set for tracking, for a given control law $u = Kx$ (LQR) (see [8] for details). Furthermore, $K$ depends on the cost weights $Q$ and $R$.

## 4. Simulation results

### 4.1. Unstable reactor

The aim of this simulation results is to compare the performance and feasibility of the proposed formulation with the classical IHMPC [7,16] and the Dual MPC that uses the LQR as terminal controller [1]. The selected system is the continuous stirred-tank reactor (CSTR) described in Ref. [24]. The objective is to control one output variable (reactor temperature) manipulating one input variable (jacket flow rate). Using Taylor series expansion and a convenient transformation, for a sampling time $T = 0.05$, the following diagonalized discrete time linear model is obtained:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1.0311 & 0 & 0 \\ 0 & 0 & 0.9559 & 0 \\ 0 & 0 & 0 & 0.5841 \end{bmatrix}, \quad B = \begin{bmatrix} 1.5524 \\ 0.7621 \\ 0.2168 \\ -0.5616 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 0.6018 & -0.9575 & -0.8965 & 0.0242 \end{bmatrix}$$

The input constraints are given by: $u_{max} = 1.32$, $u_{min} = 0$, $\Delta u_{max} = 0.1$. The input increment bounds are chosen small to clearly show its effect on the controllers' domain of attraction.

The tuning parameters for the three controllers are: $Q = 50$ and $R = 1$. Furthermore, the slack penalization for the proposed MPC is $S^i = 5 \times 10^4$. The coefficients $\nu_i$ of the generalized Minkowski functional used for Problem 2a of the proposed controller are given by: $\nu_5 = 3$, $\nu_6 = 2.5$, $\nu_7 = 2$, $\nu_8 = 1.7$, $\nu_9 = 1.5$, $\nu_{10} = 1.35$, $\nu_{11} = 1.2$, $\nu_{12} = 1.1$, $\nu_{13} = 1.03$ and $\nu_{14} = 1$ (the maximal controllable set for the unstable component is given by $C_{14}^u$).

First, we analyze the domain of attraction for the integrating-unstable components of the three controllers (assuming that for both, the classical IHMPC and Dual MPC, the desired equilibrium point is the origin). Fig. 6 shows the domain of attraction of the classical IHMPC for $m = 5$, $m = 10$, $m = 20$ and $m = 30$ (dashed-line), and the domain of attraction for the proposed MPC (solid-line). Fig. 7 shows the domain of attraction of the Dual MPC for $m = 3$, $m = 5$, $m = 10$, $m = 20$ and $m = 30$ (dashed-line), and the domain of attraction for the proposed MPC (solid-line). Notice that for the integrating-unstable modes the domain of attraction of the IHMPC and the Dual MPC tends to a limit set as $m$ increases, and furthermore, this limit set is smaller than the maximal domain of attraction of the system (i.e., the domain of attraction of the proposed controller, which does not depend on the control horizon).

Next, the performance and feasibility properties of the three controllers are analyzed. First, the system is simulated starting from a point in $C_7^{i-u}$ where the initial integrating-unstable states are close to the boundary of the maximal domain of attraction of the system. The transformed initial state is given by: $x(0) = [\,0 \quad 0.43 \quad 0.2 \quad -0.2\,]^T$, which corresponds to the following state in the original state variables as defined in Section 2: $x_1(0) = -0.13$ (concentration of reactant), $x_2(0) = 0.15$ (reactor temperature) and $x_3(0) = -0.71$ (jacket temperature), written as deviation variables. Fig. 8 shows the trajectories of the integrating-unstable states for the three controllers considered above. We can see that the proposed controller with $m = 5$ (green line) steers the system to $C_m^{i-u}$ in two time steps, and then regulates the system to the desired equilibrium point. In the same Fig. 8, it can be seen (blue-line) the state evolution for the IHMPC with $m = 30$. In the first three time steps, this controller drives the system to the set-point in a feasible way but then, the system reaches a state (shown in Fig. 8) where no

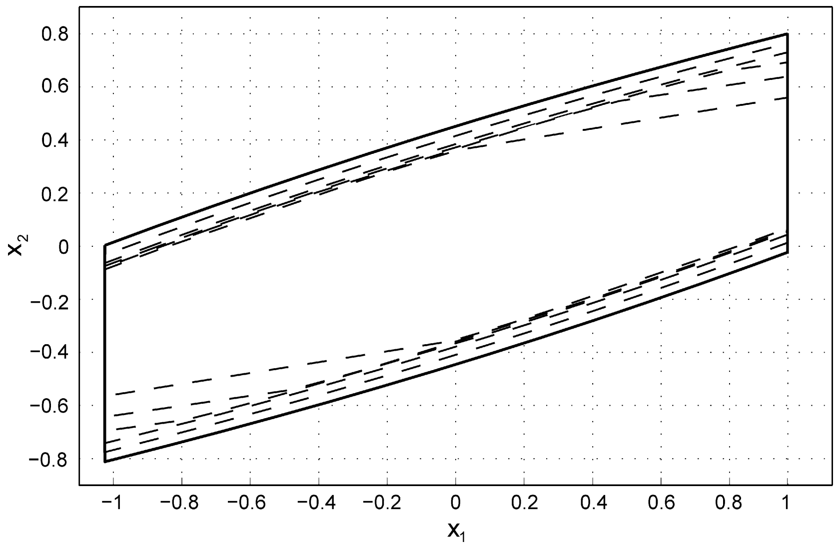


**Fig. 7.** Domain of attraction of the proposed controller (solid-line) and the Dual MPC (dashed-line).

feasible control action can be found and recursive feasibility is lost. In the same figure, the state evolution corresponding to the Dual MPC with $m = 15$ (red line) is shown. As in the conventional IHMPC, this strategy also steers the system to a point where the feasibility is lost, even with a quite large control horizon.

In a second case, a less demanding condition is simulated to compare the performances of the three controllers. The adopted control horizons are $m = 5$ for the proposed controller, $m = 17$ for the IHMPC and $m = 10$ for the Dual MPC. Fig. 9 shows the time responses when the system starts from the initial state $x(0) = [\,0.33 \quad 0 \quad 0 \quad 0\,]^T$, where the three controllers are feasible. We can see from Fig. 9 that as expected the Dual MPC controller has a slightly better performance (red line) than the others. This is so because of the optimality of the Dual MPC. However, as can be seen in the same figure, the difference between the optimal controller and the proposed controller (green line) is almost negligible. The worst performance is achieved with the conventional IHMPC (blue line), which in addition needs the largest control horizon and consequently the highest computer cost.

### 4.2. Inverted pendulum

In order to further illustrate the performance of the proposed strategy, we selected the inverted pendulum system shown in Fig. 10, where $M$ is the mass of the cart, $m$ is the mass of the pendulum, $b$ is the friction of the cart, $I$ is the inertia of the pendulum, $F$ is the force applied to the cart, $x$ is the cart position and $\theta$ is the pendulum angle from vertical (see, for instance, Sontag [25]).

We need to control both the cart's position ($y_1$) and the pendulum's angle ($y_2$), manipulating the force applied to the cart ($u$).

The corresponding state space model, in the incremental form, is given by

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1.5680 & 0 & 0 \\ 0 & 0 & 0 & 0.6387 & 0 \\ 0 & 0 & 0 & 0 & 0.91 \end{bmatrix}, \quad B = \begin{bmatrix} 0.6900 \\ -0.800 \\ 0.6314 \\ 0.2720 \\ 0.6900 \end{bmatrix}$$

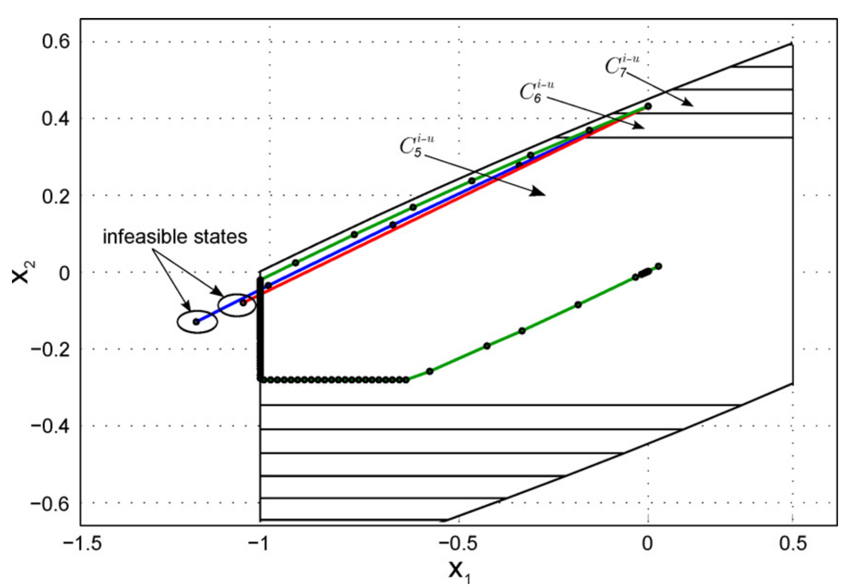


**Fig. 8.** Integrating-unstable states for the proposed controller (green line), IHMPC (blue line) and Dual MPC (red line). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of the article.)

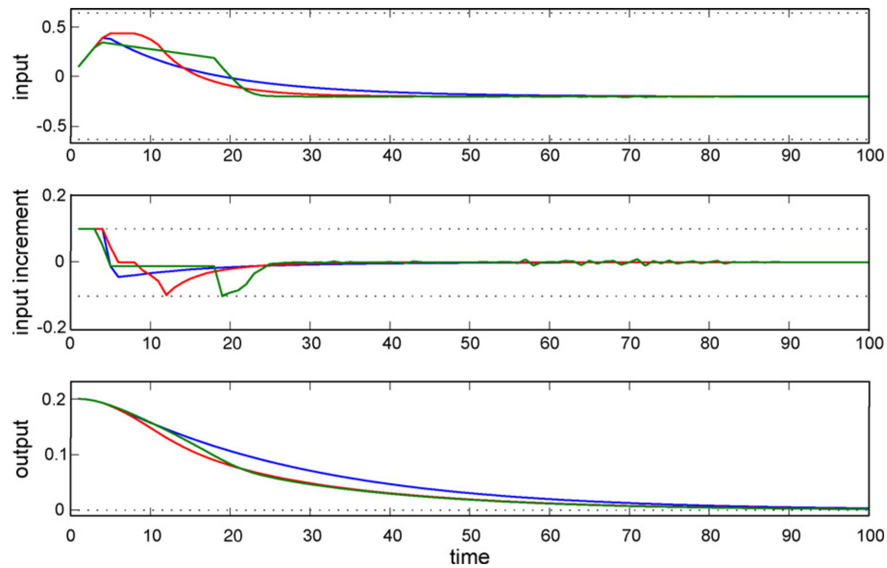


**Fig. 9.** Input, input increment and output responses for the proposed controller (green line), IHMPC (blue line) and Dual MPC (red line). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of the article.)

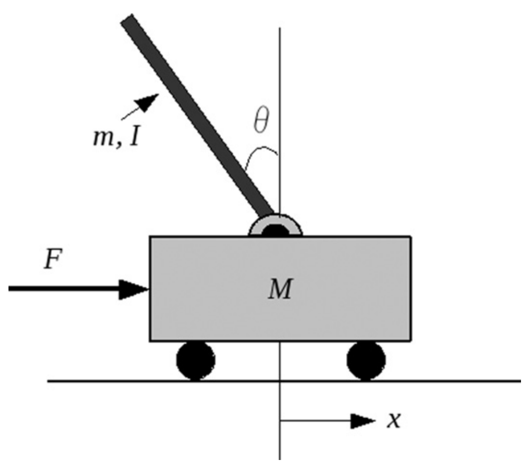


**Fig. 10.** Inverted pendulum scheme.

and

$$C = \begin{bmatrix} -1 & 0 & 0.0147 & 0.0154 & -0.99 \\ 0 & 0 & 0.1762 & 0.1750 & 0.0021 \end{bmatrix}$$

where one can easily identify two integrating modes – one being the integrating model of the original system and the other introduced by the incremental form of the model above, and one unstable mode of the original system. Clearly, the second integrating mode can be considered as a pure unstable mode, and so the system has two unstable modes and only one input (in which case the controller proposed in Ref. [11] cannot be applied). The input constraints are given by: $u_{max} = 0.5$, $u_{min} = -0.5$, $\Delta u_{max} = 0.1$, and the cart's position is constrained in the range $-0.3 \le y_1 \le 0.3$. The tuning parameters of the proposed controller are: $Q = \mathrm{diag}(10,10)$, $R = 0.1$, $m = 4$ and $S^i = 5 \times 10^4$. The coefficients $\upsilon_i$ of the generalized Minkowski functional used in Problem 2a are given by: $\upsilon_4 = 1.7$, $\upsilon_5 = 1.3$ and $\upsilon_6 = 1.1$; where for simplicity it is assumed that the maximal controllable set for the unstable modes is given by $C_7^u$, and so $\upsilon_7 = 1$. Fig. 11 shows

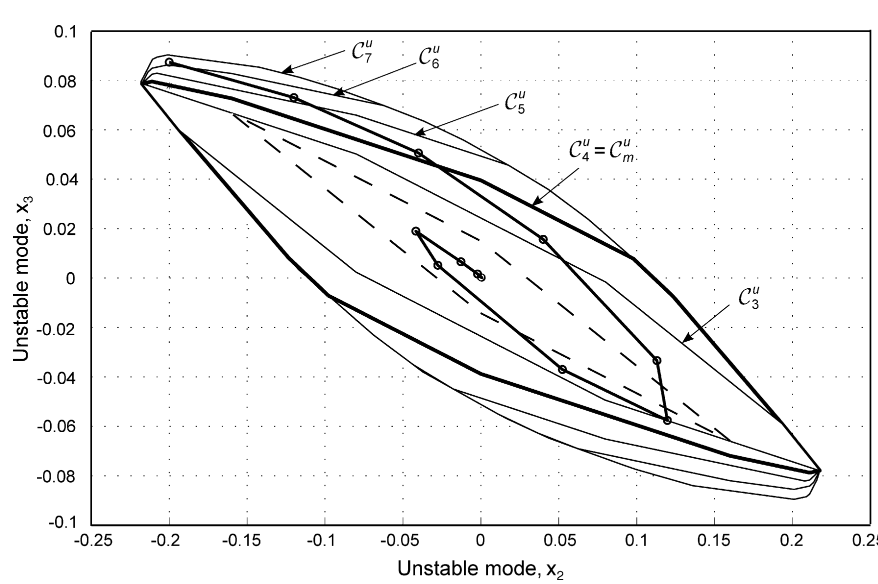


**Fig. 11.** Controllable sets for the unstable modes, and unstable state trajectory.

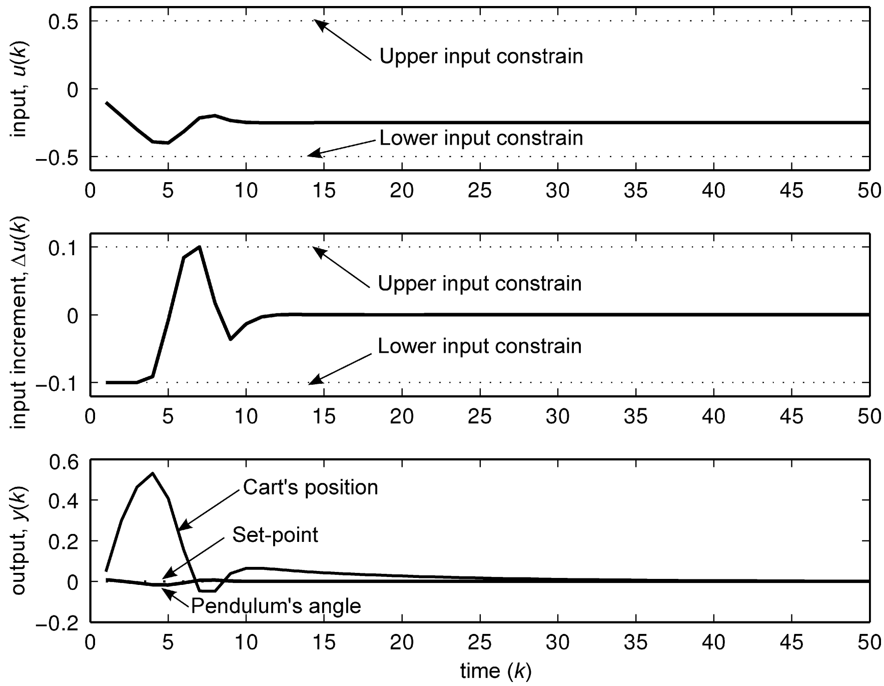


**Fig. 12.** Input, input increment and output responses.

the controllable sets for the unstable modes (given in this case by $x_2$ and $x_3$).

Initially, a state disturbance is simulated, such that the original unstable component is in $C_7^u$. In this case the controller first steers the system to $C_m^u = C_4^u$ in three steps, and then regulates the system to the equilibrium point. Fig. 11 shows the evolution of the unstable states.

In order to see how the cost of Problem 2a, $V_{a,k}$, changes for $k = 1, 2, 3, 4$, consider the corresponding values of the generalized Minkowski functional: $\Psi_{\{C_4^u,\ldots,V_7^u\}}(x^u(5|1)) = 0.35$, $\Psi_{\{C_4^u,\ldots,C_7^u\}}(x^u(6|2)) = 0.21$, $\Psi_{\{C_4^u,\ldots,C_7^u\}}(x^{un}(7|3)) = 0.05$ and $\Psi_{\{C_4^u,\ldots,C_7^u\}}(x^{un}(8|4)) = 0$, which represents a strictly decreasing sequence.

Fig. 12 shows the input, input increment and output evolution versus time. Notice that the input increment constraint becomes active at the beginning of the simulation, where the effort is made to steer the unstable part of the system to $C_m^u = C_4^u$.

In a second simulation, two set point changes are considered. In the first one the system is guided to a reachable set point given by $y_{sp} = [\,-0.2 \quad 0\,]^T$, while in the second one, the system

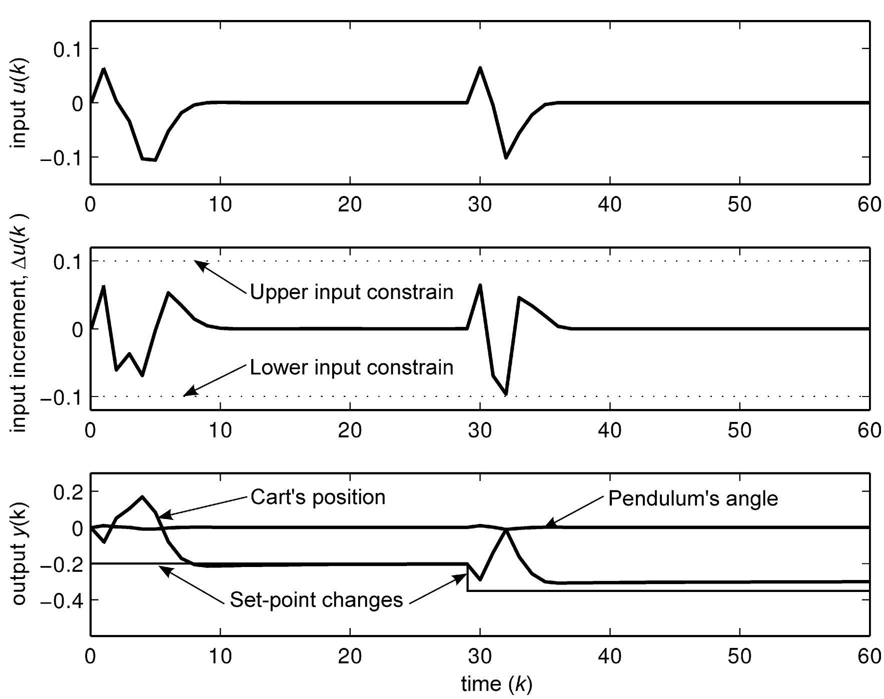


**Fig. 13.** Input, input increment and output responses for set-point changes.

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is intended to be guided to an unreachable set-point given by $y_{sp} = [\,-0.35 \quad 0\,]^T$. Fig. 13 shows the input, input increment and output evolution versus time $k$ for the two changes. In the first case the MPC cost converges to zero (the output reaches the set point), while in the second one it converges to a non-null value, maintaining the feasibility and stability. Notice that at the desired steady state the output constraint becomes active, while the input and input increments tend to zero (double integrating system).

## 5. Conclusion

A new MPC formulation was presented that exploits the properties of the model structure and a new Minkowski functional to obtain stability. As other recent formulations, it guarantees recursive feasibility and stability for tracking both, reachable and unreachable output set-points. Furthermore, in the latter case, the resulting controller steers the system to an admissible stationary output that minimizes the distance to the desired set-point, without the necessity of a target calculation stage. The main difference/benefit of the proposed approach is that it exhibits the largest possible domain of attraction (admissible set of initial states), which is a desirable characteristic for real applications. The domain of attraction of the proposed controller does not depend on the controller, but on the nature of the system, including the variable limits.

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