Convergence of an adaptive Kačanov FEM for quasi-linear problems

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Abstract

We design an adaptive finite element method to approximate the solutions of quasi-linear elliptic problems. The algorithm is based on a Kačanov iteration and a mesh adaptation step is performed after each linear solve. The method is thus *inexact* because we do not solve the discrete nonlinear problems exactly, but rather perform one iteration of a fixed point method (Kačanov), using the approximation of the previous mesh as an initial guess. The convergence of the method is proved for any *reasonable* marking strategy and starting from any initial mesh. We conclude with some numerical experiments that illustrate the theory.

Keywords: nonlinear stationary conservation laws; adaptive finite element methods; convergence.

1 Introduction

In this paper we consider quasi-linear elliptic partial differential equations over a polygonal/polyhedral domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ of the form

$$\begin{cases} -\nabla \cdot \left[\alpha(\,\cdot\,, |\nabla u|^2)\nabla u\right] = f & \text{in}\,\Omega \\ \qquad\qquad\qquad\quad u = 0 & \text{on}\,\partial\Omega, \end{cases} \tag{1}$$

where $\alpha : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ is a function whose properties will be stated in Section 2 below, and $f \in L^2(\Omega)$ is given. These equations describe stationary conservation laws which frequently arise in problems of mathematical physics [15]. For example, in hydrodynamics and gas dynamics (subsonic and supersonic flows), electrostatics, magnetostatics, heat conduction, elasticity and plasticity (e.g., plastic torsion of bars), etc. Some of these examples are better modeled by variational inequalities (see [6] and the references therein), but these fall beyond the scope of this article, which attempts to set a first step towards understanding the convergence and optimality of inexact Kačanov-type iterations, in the context of adaptive finite element methods.

For a summary of convergence and optimality results of AFEM we refer the reader to the survey [10], and the references therein. We restrict ourselves to those references strictly related to our work.

Inexact adaptive finite element methods have been considered for Stokes problem in [1, 7] using Uzawa's algorithm. Briefly, a Richardson iteration is applied to the infinite-dimensional Schur complement operator and in each iteration, the elliptic problem is solved up to a decreasing tolerance. In [1] linear convergence is proved and in [7] the optimality of the method in terms of degrees of freedom is proved, after adding some new refinement steps to the algorithm.

In [8], a contraction property is proved for an adaptive algorithm based on Dörfler's marking strategy [2] for nonlinear problems of the type (1), using Orlicz norms to cope with the very mild assumptions on the nonlinear term α.

In this work we impose stronger assumptions on α, which guarantee the convergence of Kačanov's iteration [6]. More precisely, we assume that $\alpha(\cdot,\cdot)$ is decreasing with respect to its second variable (cf. (17)), and fulfills condition (3) below; which are the same assumptions stated in [6], where they consider the iteration on a fixed space, either finite- or infinite-dimensional. Our focus is the convergence analysis of the adaptive method that results from performing one mesh adaptation in each iteration of the non-linear solver. This turns out to be a very realistic and efficient method, based on the sole assumption that a *linear* system is exactly solved in each iteration step.

This paper is organized as follows. In Section 2 we present the class of specific nonlinear problems that we study, and some of its properties. In Section 3, we present the inexact adaptive Kačanov algorithm and in Section 4 we state and prove the main result of this article, namely the convergence of the discrete solutions produced by the algorithm to the exact solution of the nonlinear problem. Finally, in Section 5, we present some numerical experiments which illustrate the theory, and explore the practical boundaries of applicability of the algorithm.

2 Setting

Let $\Omega \subset \mathbb{R}^d$ be a bounded polygonal ($d = 2$) or polyhedral ($d = 3$) domain with Lipschitz boundary. A weak formulation of (1) consists in finding $u \in H_0^1(\Omega)$ such that

$$a(u; u, v) = L(v), \qquad \forall\, v \in H_0^1(\Omega), \tag{2}$$

where

$$a(w; u, v) = \int_\Omega \alpha(\,\cdot\,, |\nabla w|^2)\nabla u \cdot \nabla v, \qquad \forall\, w, u, v \in H_0^1(\Omega),$$

and

$$L(v) = \int_\Omega f v, \qquad \forall\, v \in H_0^1(\Omega).$$

We require that α is locally Lipschitz in its first variable, uniformly with respect to its second variable (cf. (15) below). On the other hand, we assume that α is $\mathcal{C}^1$ as a function of its second variable and there exist positive constants c_a and C_a such that

$$c_a \le \alpha(x, t^2) + 2t^2 D_2\alpha(x, t^2) \le C_a, \qquad \forall\, x \in \Omega,\ t > 0, \tag{3}$$

where $D_2\alpha$ denotes the partial derivative of α with respect to its second variable. The last condition is related to the well-posedness of problem (2) as we will show below. Also, it is possible to prove that (3) implies that

$$c_a \le \alpha(x, t) \le C_a, \qquad \forall\, x \in \Omega,\ t > 0. \tag{4}$$

These same assumptions are stated in [6], where several interesting applied problems are shown to satisfy them.

Additionally, it is easy to check that the form a is linear and symmetric in its second and third variables. Also, from (4) it follows that a is bounded,

$$|a(w;u,v)| \leq C_a \|\nabla u\|_\Omega \|\nabla v\|_\Omega, \qquad \forall\, w,u,v \in H_0^1(\Omega), \tag{5}$$

and coercive,

$$c_a \|\nabla u\|_\Omega^2 \leq a(w;u,u), \qquad \forall\, w,u \in H_0^1(\Omega). \tag{6}$$

If we define $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ as the nonlinear operator given by

$$\langle Au, v\rangle := a(u;u,v), \qquad \forall\, u,v \in H_0^1(\Omega),$$

then problem (2) is equivalent to the equation

$$Au = L,$$

where $L \in H^{-1}(\Omega)$ is given. Assumption (3) implies that A is Lipschitz and strongly monotone (see [15]), i.e., there exist positive constants C_A and c_A such that

$$\|Au - Av\|_{H^{-1}(\Omega)} \leq C_A \|\nabla(u-v)\|_\Omega, \qquad \forall\, u,v \in H_0^1(\Omega), \tag{7}$$

and

$$\langle Au - Av, u - v\rangle \geq c_A \|\nabla(u-v)\|_\Omega^2, \qquad \forall\, u,v \in H_0^1(\Omega). \tag{8}$$

As a consequence of (7) and (8), problem (2) has a unique solution and is stable [14, 15].

3 Adaptive algorithm

In order to define discrete adaptive approximations to problem (2) we will consider *triangulations* of the domain Ω. Let $\mathcal{T}_0$ be an initial conforming triangulation of Ω, that is, a partition of Ω into d-simplices such that if two elements intersect, they do so at a full vertex/edge/face of both elements. Let $\mathbb{T}$ denote the set of all conforming triangulations of Ω obtained from $\mathcal{T}_0$ by refinement using the bisection procedures presented by Stevenson [13]. These coincide (after some re-labeling) with the so-called *newest vertex* bisection procedure in two dimensions and the bisection procedure of Kossaczký in three dimensions [11].

Due to the processes of refinement used, the family $\mathbb{T}$ is shape regular, i.e.,

$$\sup_{\mathcal{T}\in\mathbb{T}} \sup_{T\in\mathcal{T}} \frac{\operatorname{diam}(T)}{\rho_T} =: \kappa_{\mathbb{T}} < \infty,$$

where $\operatorname{diam}(T)$ is the diameter of T, and ρ_T is the radius of the largest ball contained in it. Throughout this article, we only consider meshes $\mathcal{T}$ that belong to the family $\mathbb{T}$, so the shape regularity of all of them is bounded by the uniform constant $\kappa_{\mathbb{T}}$ which only depends on the initial triangulation $\mathcal{T}_0$ [11]. Also, the diameter of any element $T \in \mathcal{T}$ is equivalent to the local mesh-size $H_T := |T|^{1/d}$, which in turn defines the global mesh-size $H_{\mathcal{T}} := \max_{T\in\mathcal{T}} H_T$.

For the discretization we consider the Lagrange finite element spaces consisting of continuous functions vanishing on $\partial\Omega$ which are piecewise linear over a mesh $\mathcal{T} \in \mathbb{T}$, i.e.,

$$\mathbb{V}_{\mathcal{T}} := \{v \in H_0^1(\Omega) \mid \quad v_{|_T} \in \mathcal{P}_1(T), \quad \forall\, T \in \mathcal{T}\}. \tag{9}$$

We are now in position to state the adaptive loop to approximate the solution u of the problem (2).

Adaptive Algorithm. `Let` $\mathcal{T}_0$ `be an initial conforming mesh of` Ω `and` $u_0 \in \mathbb{V}_{\mathcal{T}_0}$ `be an initial approximation. Let` $\mathcal{T}_1 = \mathcal{T}_0$ `and` $k = 1$.

1. $u_k := \mathsf{SOLVE}(u_{k-1}, \mathcal{T}_k)$.
2. $\{\eta_k(T)\}_{T\in\mathcal{T}_k} := \mathsf{ESTIMATE}(u_{k-1}, u_k, \mathcal{T}_k)$.
3. $\mathcal{M}_k := \mathsf{MARK}(\{\eta_k(T)\}_{T\in\mathcal{T}_k}, \mathcal{T}_k)$.
4. $\mathcal{T}_{k+1} := \mathsf{REFINE}(\mathcal{T}_k, \mathcal{M}_k, n)$.
5. `Increment` k `and go back to step 1.`

We now explain each module of the last algorithm in detail.

The module SOLVE. Given the conforming triangulation $\mathcal{T}_k$ of Ω, and the solution u_{k-1} from the previous iteration, the module SOLVE outputs the solution $u_k \in \mathbb{V}_k := \mathbb{V}_{\mathcal{T}_k}$ of the *linear* problem

$$a(u_{k-1}; u_k, v_k) = L(v_k), \qquad \forall\, v_k \in \mathbb{V}_k. \tag{10}$$

Remark 3.1 (Linear vs. nonlinear)**.** Notice that while SOLVE requires the solution of a linear system, the usual discretization of (2) in $\mathbb{V}_k$ consists in finding $u_k \in \mathbb{V}_k$ such that

$$a(u_k; u_k, v_k) = L(v_k), \qquad \forall\, v_k \in \mathbb{V}_k, \tag{11}$$

which is nonlinear. We propose here to make only one step of a fixed point iterative method to solve the nonlinear problem, and proceed to the mesh adaptation, whereas the usual approach would be to approximate the discrete nonlinear problem up to a very fine accuracy (pretending to have it exactly solved) before proceeding to mesh adaptation [5]. Each iteration of the adaptive loop is thus much cheaper in our approach. Our theory guarantees convergence of this algorithm, and the numerical experiments of Section 5 show that the convergence is quasi-optimal, although the latter is not yet rigorously proved.

Remark 3.2 (Kačanov's Method)**.** Let us consider problem (2) (resp. problem (11) with $k \in \mathbb{N}$ fixed). We denote the space $H_0^1(\Omega)$ (resp. $\mathbb{V}_k$) by $\mathbb{V}$, and the solution u (resp. u_k) by U. Given an initial approximation $U_0 \in \mathbb{V}$ of the solution U, we consider the sequence $\{U_i\}_{i\in\mathbb{N}_0}$ where $U_i \in \mathbb{V}$ is the solution of the *linear* problem

$$a(U_{i-1}; U_i, v) = L(v), \qquad \forall\, v \in \mathbb{V}, \quad i \in \mathbb{N}.$$

This is known as Kačanov's Method, and it follows [6] that the sequence $\{U_i\}_{i\in\mathbb{N}_0}$ converges to the solution U, provided $D_2\alpha(x,t) \le 0$ for all $x \in \Omega$ and $t > 0$.

Notice that our algorithm consists in performing *only one* step of Kačanov iteration in each step of the adaptive loop.

The module ESTIMATE. Given $\mathcal{T}_k$, u_{k-1} and the corresponding output u_k of SOLVE, the module ESTIMATE computes and outputs the local error estimators $\{\eta_k(T)\}_{T\in\mathcal{T}_k}$ given by

$$\eta_k^2(T) := H_T^2 \, \|R_k\|_T^2 + H_T \, \|J_k\|_{\partial T}^2 \,, \tag{12}$$

for all $T \in \mathcal{T}_k$. Here R_k denotes the *element residual* given by

$$R_{k|_T} := -\nabla \cdot \left[\alpha(\,\cdot\,, |\nabla u_{k-1}|^2)\nabla u_k\right] - f, \qquad \forall\, T \in \mathcal{T}_k,$$

and J_k denotes the *jump residual* given by

$$J_{k|_S} := \frac{1}{2}\left[(\alpha(\,\cdot\,, |\nabla u_{k-1}|^2)\nabla u_k)_{|_T} \cdot \vec{n} + (\alpha(\,\cdot\,, |\nabla u_{k-1}|^2)\nabla u_k)_{|_{T'}} \cdot \vec{n}'\right],$$

for each interior side S, and $J_{k|_S} := 0$, if S is a side lying on the boundary of Ω. Here, T and T' denote the elements of $\mathcal{T}_k$ sharing S, and $\vec{n}$, $\vec{n}'$ are the outward unit normals of T, T' on S, respectively.

The squared sum of the a posteriori error estimators is an upper bound for the *residual* $\mathbf{R}(u_k) \in H^{-1}(\Omega)$ of u_k which is defined as

$$\langle \mathbf{R}(u_k), v\rangle := a(u_{k-1}; u_k, v) - L(v) = \int_\Omega \left(\alpha(\,\cdot\,, |\nabla u_{k-1}|^2)\nabla u_k \cdot \nabla v - fv\right),$$

for $v \in H_0^1(\Omega)$. In fact, integrating by parts on each $T \in \mathcal{T}_k$ we have that

$$\langle \mathbf{R}(u_k), v\rangle = \sum_{T\in\mathcal{T}_k} \left(\int_T R_k v + \int_{\partial T} J_k v\right), \tag{13}$$

and since $\langle \mathbf{R}(u_k), v_k\rangle = 0$ for $v_k \in \mathbb{V}_k$, using (13) and interpolation estimates, we arrive at the following upper bound:[1]

$$|\langle \mathbf{R}(u_k), v\rangle| \lesssim \sum_{T\in\mathcal{T}_k} \eta_k(T) \|\nabla v\|_{\omega_k(T)}, \quad \forall\, v \in H_0^1(\Omega), \tag{14}$$

where $\omega_k(T)$ is the union of T and its neighbors in $\mathcal{T}_k$.

The next result is some kind of *stability* result for the estimators, which is a property somewhat weaker than the usual *efficiency*, but sufficient to guarantee convergence of our adaptive algorithm (see also [12, 4]). In order to prove it, we assume that α is locally Lipschitz in its first variable, uniformly with respect to its second variable, as we mentioned before. More precisely, we assume that there exists a constant $C_\alpha > 0$ such that

$$|\alpha(x,t) - \alpha(y,t)| \le C_\alpha |x - y|, \qquad \forall\, x, y \in T,\ t > 0, \tag{15}$$

for each $T \in \mathcal{T}_0$.

[1] From now on, we will write $a \lesssim b$ to indicate that $a \le Cb$ with $C > 0$ a constant depending on the data of the problem and possibly on shape regularity $\kappa_{\mathbb{T}}$ of the meshes.

Proposition 3.3 (Stability of the local error estimators)**.** *Let $\{u_k\}_{k\in\mathbb{N}}$ be the sequence of discrete solutions computed with the Adaptive Algorithm. Then, the local error estimators given by* (12) *are stable. More precisely, there holds*

$$\eta_k(T) \lesssim \|\nabla u_k\|_{\omega_k(T)} + \|f\|_T, \qquad \forall\, T \in \mathcal{T}_k,$$

for all $k \in \mathbb{N}$.

Proof. Let $\{u_k\}_{k\in\mathbb{N}}$ be the sequence computed with the Adaptive Algorithm. Let $k \in \mathbb{N}$ and $T \in \mathcal{T}_k$ be fixed. On the one hand, using that $u_{k|T}$ is linear, and thus $\Delta u_k = 0$ inside T, we have that

$$\begin{aligned}\|R_k\|_T &= \left\| -\nabla \cdot \left[\alpha(\cdot, |\nabla u_{k-1}|^2)\nabla u_k\right] - f \right\|_T \\ &\leq \left\| \nabla\left[\alpha(\cdot, |\nabla u_{k-1}|^2)\right] \cdot \nabla u_k \right\|_T + \|f\|_T\,.\end{aligned}$$

Since α is locally Lipschitz in its first variable (cf. (15)), it follows that if $\xi := \nabla u_{k-1}$ (constant over T),

$$\left| \frac{\partial}{\partial x_i} \alpha(x, |\xi|^2) \right| = \left| \frac{\partial \alpha}{\partial x_i}(x, |\xi|^2) \right| \leq C_\alpha, \qquad \forall\, x \in T, \quad 1 \leq i \leq d,$$

and thus,

$$\|R_k\|_T \lesssim \|\nabla u_k\|_T + \|f\|_T\,.$$

On the other hand, if S is a side of $\mathcal{T}_k$ shared by the elements $T, T' \in \mathcal{T}_k$,

$$\begin{aligned}\|J_k\|_S &\leq \left\| (\alpha(\,\cdot\,, |\nabla u_{k-1}|^2)\nabla u_k)_{|T} \right\|_S + \left\| (\alpha(\,\cdot\,, |\nabla u_{k-1}|^2)\nabla u_k)_{|T'} \right\|_S \\ &\lesssim \left\| \nabla u_{k|_T} \right\|_S + \left\| \nabla u_{k|_{T'}} \right\|_S \\ &\lesssim H_T^{-1/2} \|\nabla u_k\|_T + H_{T'}^{-1/2} \|\nabla u_k\|_{T'} \lesssim H_T^{-1/2} \|\nabla u_k\|_{T\cup T'},\end{aligned}$$

where we have used (4) and a scaled trace theorem. Therefore,

$$H_T^{1/2}\, \|J_k\|_{\partial T} \lesssim \|\nabla u_k\|_{\omega_k(T)},$$

which completes the proof. $\square$

The module MARK. Based on the local error indicators, the module MARK selects a subset $\mathcal{M}_k$ of $\mathcal{T}_k$, using any of the so-called *reasonable* marking strategies, such as the *maximum strategy*, the *equidistribution strategy*, or *Dörfler's strategy* [11]. More precisely, we only require that the set of marked elements $\mathcal{M}_k$ has at least one element of $\mathcal{T}_k$ holding the largest local estimator. That is, there exists an element $T_k^{\max} \in \mathcal{M}_k$ such that

$$\eta_k(T_k^{\max}) = \max_{T\in\mathcal{T}_k} \eta_k(T). \tag{16}$$

This is what practitioners usually do in order to maximize the error reduction with a minimum computational effort.

The module REFINE. Finally, the module REFINE takes the mesh $\mathcal{T}_k$ and the subset $\mathcal{M}_k \subset \mathcal{T}_k$ as inputs. By using the bisection rule described by Stevenson in [13], this module refines (bisects) each element in $\mathcal{M}_k$ at least n times (where $n \geq 1$ is fixed), in order to obtain a new conforming triangulation $\mathcal{T}_{k+1}$ of Ω, which is a refinement of $\mathcal{T}_k$ and the output of this module. By definition, $\mathcal{T}_k \in \mathbb{T}$ for all $k \in \mathbb{N}$ and the family of meshes obtained by the Adaptive Algorithm is shape-regular. Finally, it is worth observing that the resulting spaces are nested, i.e., $\mathbb{V}_k \subset \mathbb{V}_{k+1}$; this fact will be used in the proof of Lemma 4.2 below.

Remark 3.4 (The adaptive sequence $\{u_k\}_{k\in\mathbb{N}_0}$ is bounded)**.** By the coercivity of a given by (6), and using the definition (10) of u_k we have that

$$\|\nabla u_k\|_\Omega^2 \leq \frac{1}{c_a} a(u_{k-1}; u_k, u_k) = \frac{1}{c_a} L(u_k) \leq \frac{\|L\|_{H^{-1}(\Omega)}}{c_a} \|\nabla u_k\|_\Omega,$$

for all $k \in \mathbb{N}$, and then

$$\|\nabla u_k\|_\Omega \leq \frac{\|L\|_{H^{-1}(\Omega)}}{c_a}.$$

Therefore, $\{u_k\}_{k\in\mathbb{N}_0}$ is bounded in $H_0^1(\Omega)$.

4 Convergence Analysis

In this section we prove the convergence of the sequence computed with the Adaptive Algorithm described in the previous section. We combine the ideas of the proof of convergence of adaptive algorithms for linear problems given in [9] and [12] with new techniques needed to overcome the difficulties arisen by the nonlinear nature of the problem treated in this paper, adapting some ideas from [3, 4].

As a first step to prove the convergence of the $\{u_k\}_{k\in\mathbb{N}_0}$ to the exact solution u, we prove that $\|\nabla(u_k - u_{k+1})\|_\Omega \to 0$, as k tends to infinity, for which we need the following auxiliary lemma.

From now on we assume that

$$\alpha(x, t_1) \geq \alpha(x, t_2) \text{ whenever } 0 \leq t_1 \leq t_2 \text{ and } x \in \Omega. \tag{17}$$

Lemma 4.1. *Let us assume that $\alpha(x,\cdot)$ is a monotone decreasing function for all $x \in \Omega$, i.e., (17) holds. Then*

$$\mathcal{J}(v) - \mathcal{J}(w) \leq \frac{1}{2}\big(a(w; v, v) - a(w; w, w)\big), \qquad \forall\, v, w \in H_0^1(\Omega),$$

where $\mathcal{J}(v) = \int_0^1 a(sv; sv, v)\, ds$.

Proof. Let $v, w \in H_0^1(\Omega)$. Then, a change of variables in the integral defining $\mathcal{J}(\cdot)$ yields

$$\begin{aligned}\mathcal{J}(v) - \mathcal{J}(w) &= \frac{1}{2} \int_\Omega \left(\int_0^{|\nabla v|^2} \alpha(x,t)\, dt - \int_0^{|\nabla w|^2} \alpha(x,t)\, dt \right) dx \\ &= \frac{1}{2} \int_\Omega \int_{|\nabla w|^2}^{|\nabla v|^2} \alpha(x,t)\, dt\, dx.\end{aligned}$$

Since $\alpha(x,\cdot)$ is a decreasing function for all $x \in \Omega$,

$$\begin{aligned}\frac{1}{2}\int_\Omega \int_{|\nabla w|^2}^{|\nabla v|^2} \alpha(x,t)\ dt\ dx &\le \frac{1}{2}\int_\Omega \alpha(x,|\nabla w|^2)\left(|\nabla v|^2 - |\nabla w|^2\right)\ dx \\ &= \frac{1}{2}\big(a(w;v,v) - a(w;w,w)\big),\end{aligned}$$

and the assertion of the lemma follows. □

Lemma 4.2. *Let $\{u_k\}_{k\in\mathbb{N}_0}$ denote the sequence of discrete solutions computed with the Adaptive Algorithm. Then,*

$$\lim_{k\to\infty} \|\nabla(u_k - u_{k+1})\|_\Omega = 0.$$

Proof. Let $\{u_k\}_{k\in\mathbb{N}_0}$ be the sequence obtained with the Adaptive Algorithm. Since a is coercive (cf. (6)), and linear and symmetric in its second and third variables, we have that

$$\begin{aligned}c_a\|\nabla(u_k - u_{k+1})\|_\Omega^2 &\le a(u_k; u_k - u_{k+1}, u_k - u_{k+1}) \\ &= a(u_k;u_k,u_k) - 2a(u_k;u_{k+1},u_k) + a(u_k;u_{k+1},u_{k+1}).\end{aligned} \tag{18}$$

From (10), since $u_k \in \mathbb{V}_{k+1}$, it follows that $a(u_k;u_{k+1},u_k - u_{k+1}) = L(u_k - u_{k+1})$ and thus

$$a(u_k;u_{k+1},u_k) = L(u_k - u_{k+1}) + a(u_k;u_{k+1},u_{k+1}).$$

Replacing this equality in (18) and taking into account Lemma 4.1 we obtain

$$\begin{aligned}c_a\|\nabla(u_k - u_{k+1})\|_\Omega^2 &\le a(u_k;u_k,u_k) - 2L(u_k - u_{k+1}) - a(u_k;u_{k+1},u_{k+1}) \\ &\le 2\mathcal{J}(u_k) - 2\mathcal{J}(u_{k+1}) - 2L(u_k) + 2L(u_{k+1}) \\ &= 2(\mathcal{F}(u_k) - \mathcal{F}(u_{k+1})),\end{aligned} \tag{19}$$

where $\mathcal{F} := \mathcal{J} - L$, and therefore, $\{\mathcal{F}(u_k)\}_{k\in\mathbb{N}_0}$ is a monotone decreasing sequence.

On the other hand, $\{\mathcal{F}(u_k)\}_{k\in\mathbb{N}_0}$ is bounded below since

$$\begin{aligned}\mathcal{F}(u_k) &= \int_0^1 s a(su_k;u_k,u_k)\ ds - L(u_k) \\ &\ge \frac{1}{2}c_a\|\nabla u_k\|_\Omega^2 - \|L\|_{H^{-1}(\Omega)}\|\nabla u_k\|_\Omega \ge -\frac{\|L\|_{H^{-1}(\Omega)}^2}{2c_a}.\end{aligned}$$

Finally, from the last two assertions it follows that $\{\mathcal{F}(u_k)\}_{k\in\mathbb{N}_0}$ is convergent. Considering (19), we conclude the proof of this lemma. □

We show now that the sequence obtained with the Adaptive Algorithm is convergent, and more precisely, that it converges to a function in the limiting space $\mathbb{V}_\infty := \overline{\cup\mathbb{V}_k}^{H_0^1(\Omega)}$. Note that $\mathbb{V}_\infty$ is a Hilbert space with the inner product inherited from $H_0^1(\Omega)$.

Theorem 4.3 (The adaptive sequence is convergent)**.** *Let $\{u_k\}_{k\in\mathbb{N}_0}$ be the sequence obtained with the Adaptive Algorithm. Let $u_\infty \in \mathbb{V}_\infty$ be the only solution to*

$$u_\infty \in \mathbb{V}_\infty : \quad a(u_\infty;u_\infty,v) = L(v), \qquad \forall\ v \in \mathbb{V}_\infty. \tag{20}$$

Then

$$u_k \longrightarrow u_\infty \quad \textit{in } H_0^1(\Omega).$$

Remark 4.4. Notice that (20) always has a solution because (7) and (8) hold on $\mathbb{V}_\infty$, which is itself a Hilbert space.

Proof. Let $\{u_k\}_{k\in\mathbb{N}_0}$ be the sequence obtained with the Adaptive Algorithm. Let $u_\infty \in \mathbb{V}_\infty$ denote the solution of (20) and $\mathcal{P}_{k+1} : H_0^1(\Omega) \to \mathbb{V}_{k+1}$ be the orthogonal projection onto $\mathbb{V}_{k+1}$. Since A is strongly monotone (cf. (8)), using (20) and (10) we have that

$$\begin{aligned}
c_A\|\nabla(u_k - u_\infty)\|_\Omega^2 &\le \langle Au_k - Au_\infty, u_k - u_\infty\rangle \\
&= \langle Au_k, u_k - u_\infty\rangle - L(u_k - u_\infty) \\
&= \langle Au_k, u_k - \mathcal{P}_{k+1}u_\infty\rangle + \langle Au_k, \mathcal{P}_{k+1}u_\infty - u_\infty\rangle \\
&\quad - L(u_k - \mathcal{P}_{k+1}u_\infty) - L(\mathcal{P}_{k+1}u_\infty - u_\infty) \\
&= a(u_k; u_k - u_{k+1}, u_k - \mathcal{P}_{k+1}u_\infty) + \langle Au_k, \mathcal{P}_{k+1}u_\infty - u_\infty\rangle \\
&\quad - L(\mathcal{P}_{k+1}u_\infty - u_\infty) \\
&\le C_a\|\nabla(u_k - u_{k+1})\|_\Omega(\|\nabla u_k\|_\Omega + \|\nabla u_\infty\|_\Omega) \\
&\quad + (C_a\|\nabla u_k\|_\Omega + \|L\|_{H^{-1}(\Omega)})\|\nabla(\mathcal{P}_{k+1}u_\infty - u_\infty)\|_\Omega,
\end{aligned}$$

for all $k \in \mathbb{N}_0$, where in the last inequality we have used (5). From Remark 3.4 it follows that $\{u_k\}_{k\in\mathbb{N}_0}$ is bounded in $H_0^1(\Omega)$, and using Lemma 4.2, together with the fact that the spaces $\{\mathbb{V}_k\}_{k\in\mathbb{N}_0}$ are nested and $\cup_{k\in\mathbb{N}_0}\mathbb{V}_k$ is dense in $\mathbb{V}_\infty$, we conclude that $u_k \to u_\infty$ in $H_0^1(\Omega)$. ◻

In order to show that the limiting function u_∞ is, in fact, the solution of the problem (2) and thereby conclude that the adaptive sequence converges to the solution of this problem, we establish first two auxiliary results (see Lemma 4.6 and Theorem 4.7). We need the following

Definition 4.5. Given any sequence of meshes $\{\mathcal{T}_k\}_{k\in\mathbb{N}_0} \subset \mathbb{T}$, with $\mathcal{T}_{k+1}$ a refinement of $\mathcal{T}_k$, for each $k \in \mathbb{N}_0$, we define

$$\mathcal{T}_k^+ := \{T \in \mathcal{T}_k \mid T \in \mathcal{T}_m, \quad \forall\, m \ge k\}, \qquad \mathcal{T}_k^0 := \mathcal{T}_k \setminus \mathcal{T}_k^+,$$

and

$$\Omega_k^+ := \bigcup_{T\in\mathcal{T}_k^+} \omega_k(T), \qquad \Omega_k^0 := \bigcup_{T\in\mathcal{T}_k^0} \omega_k(T).$$

In words, $\mathcal{T}_k^+$ is the subset of the elements of $\mathcal{T}_k$ which are never refined in the adaptive process, and $\mathcal{T}_k^0$ consists of the elements which are eventually refined.

It can be proved [9] that if $\chi_{\Omega_k^0}$ denotes the characteristic function of Ω_k^0, then

$$\lim_{k\to\infty} \|h_k\chi_{\Omega_k^0}\|_{L^\infty(\Omega)} = 0, \tag{21}$$

where $h_k \in L^\infty(\Omega)$ denotes the piecewise constant mesh-size function satisfying ${h_k}_{|_T} := H_T$, for all $T \in \mathcal{T}_k$.

Since the error estimators are stable (cf. Proposition 3.3), using the convergence proved in the last theorem and (21) we can establish the following

Lemma 4.6 (Estimator on marked elements)**.** *Let $\{\{\eta_k(T)\}_{T\in\mathcal{T}_k}\}_{k\in\mathbb{N}_0}$ be the sequence of local error estimators computed with the Adaptive Algorithm, and* *let $\{\mathcal{M}_k\}_{k\in\mathbb{N}_0}$ be the sequence of subsets of marked elements over each mesh. Then,*

$$\lim_{k\to\infty} \max_{T\in\mathcal{M}_k} \eta_k(T) = 0.$$

Proof. Let $\{\{\eta_k(T)\}_{T\in\mathcal{T}_k}\}_{k\in\mathbb{N}_0}$ and $\{\mathcal{M}_k\}_{k\in\mathbb{N}_0}$ be as in the assumptions. For each $k \in \mathbb{N}_0$, we select $T_k \in \mathcal{M}_k$ such that $\eta_k(T_k) = \max_{T\in\mathcal{M}_k} \eta_k(T)$. Using Proposition 3.3 we have that

$$\eta_k(T_k) \lesssim \|\nabla u_k\|_{\omega_k(T_k)} + \|f\|_{T_k} \lesssim \|\nabla u_k - \nabla u_\infty\|_\Omega + \|\nabla u_\infty\|_{\omega_k(T_k)} + \|f\|_{T_k}, \quad (22)$$

where $u_\infty \in \mathbb{V}_\infty$ is the solution of (20). On the one hand, the first term in the right hand side of (22) tends to zero due to Theorem 4.3. On the other hand, since $T_k \in \mathcal{M}_k \subset \mathcal{T}_k^0$, from (21) it follows that

$$|T_k| \le |\omega_k(T_k)| \lesssim H_{T_k}^d \le \|h_k \chi_{\Omega_k^0}\|_{L^\infty(\Omega)}^d \to 0, \quad \text{as } k \to \infty,$$

and the last two terms in the right hand side of (22) also tend to zero. □

Using the upper bound (14), the stability of the estimators (Proposition 3.3), the facts that $\{u_k\}_{k\in\mathbb{N}_0}$ is bounded (cf. Remark 3.4) and the marking strategy is reasonable (cf. (16)), and Lemma 4.6, we now prove the following important result.

Theorem 4.7 (Weak convergence of the residual)**.** *If $\{u_k\}_{k\in\mathbb{N}_0}$ denotes the sequence of discrete solutions computed with the Adaptive Algorithm, then*

$$\lim_{k\to\infty} \langle \mathbf{R}(u_k), v\rangle = 0, \quad \textit{for all } v \in H_0^1(\Omega).$$

Proof. We prove first the result for $v \in H^2(\Omega) \cap H_0^1(\Omega)$, and then extend it to $H_0^1(\Omega)$ by density. Let $p \in \mathbb{N}$ and $k > p$. By Definition 4.5 we have that $\mathcal{T}_p^+ \subset \mathcal{T}_k^+ \subset \mathcal{T}_k$. Let $v_k \in \mathbb{V}_k$ be the Lagrange's interpolant of v. Since $\langle \mathbf{R}(u_k), v_k\rangle = 0$, using (14), and Cauchy-Schwartz's inequality we have that

$$\begin{aligned}
|\langle \mathbf{R}(u_k), v\rangle| = |\langle \mathbf{R}(u_k), v - v_k\rangle| &\lesssim \sum_{T\in\mathcal{T}_k} \eta_k(T)\|\nabla(v-v_k)\|_{\omega_k(T)} \\
&= \sum_{T\in\mathcal{T}_p^+} \eta_k(T)\|\nabla(v-v_k)\|_{\omega_k(T)} + \sum_{T\in\mathcal{T}_k\setminus\mathcal{T}_p^+} \eta_k(T)\|\nabla(v-v_k)\|_{\omega_k(T)} \\
&\lesssim \eta_k(\mathcal{T}_p^+)\|\nabla(v-v_k)\|_{\Omega_p^+} + \eta_k(\mathcal{T}_k\setminus\mathcal{T}_p^+)\|\nabla(v-v_k)\|_{\Omega_p^0},
\end{aligned}$$

where, for any $\Xi \subset \mathcal{T}_k$ we hereafter denote $\left(\sum_{T\in\Xi} \eta_k^2(T)\right)^{\frac{1}{2}}$ by $\eta_k(\Xi)$. Taking into account Proposition 3.3 and the boundedness of the discrete solutions (cf. Remark 3.4) we have that $\eta_k(\mathcal{T}_k \setminus \mathcal{T}_p^+) \le \eta_k(\mathcal{T}_k) \lesssim 1$, and therefore,

$$|\langle \mathbf{R}(u_k), v\rangle| \lesssim \left(\eta_k(\mathcal{T}_p^+) + \|h_p\chi_{\Omega_p^0}\|_{L^\infty(\Omega)}\right) |v|_{H^2(\Omega)},$$

due to interpolation estimates.

In order to prove that $\langle \mathbf{R}(u_k), v\rangle \to 0$ as $k \to \infty$ we now let $\varepsilon > 0$ be arbitrary. Due to (21), there exists $p \in \mathbb{N}$ such that

$$\|h_p\chi_{\Omega_p^0}\|_{L^\infty(\Omega)} < \varepsilon.$$

On the other hand, since $\mathcal{T}_p^+ \subset \mathcal{T}_k^+ \subset \mathcal{T}_k$ and the marking strategy is reasonable (cf. (16)),

$$\eta_k(\mathcal{T}_p^+) \le (\#\mathcal{T}_p^+)^{1/2} \max_{T\in\mathcal{T}_p^+} \eta_k(T) \le (\#\mathcal{T}_p^+)^{1/2} \max_{T\in\mathcal{M}_k} \eta_k(T).$$

Now, by Lemma 4.6, we can select $K > p$ such that $\eta_k(\mathcal{T}_p^+) < \varepsilon$, for all $k > K$.

Summarizing, we have proved that

$$\lim_{k\to\infty} \langle \mathbf{R}(u_k), v\rangle = 0, \qquad \text{for all } v \in H^2(\Omega) \cap H_0^1(\Omega).$$

Finally, since $H^2(\Omega) \cap H_0^1(\Omega)$ is dense in $H_0^1(\Omega)$, this limit is also zero for all $v \in H_0^1(\Omega)$. □

As a consequence of Theorem 4.7 we now prove that u_∞ is the solution of problem (2).

Theorem 4.8 (The limiting function is the solution)**.** *If u_∞ denotes the solution of (20), then u_∞ is the solution of problem (2), i.e.,*

$$a(u_\infty; u_\infty, v) = L(v), \qquad \forall\, v \in H_0^1(\Omega).$$

Proof. Let u_∞ be the solution of (20). If $v \in H_0^1(\Omega)$, and $\{u_k\}_{k\in\mathbb{N}_0}$ denotes the sequence of discrete solutions computed with the Adaptive Algorithm, then

$$\begin{aligned}
|a(u_\infty; u_\infty, v) - L(v)| &= |a(u_\infty; u_\infty, v) - L(v) - a(u_k; u_{k+1}, v) + a(u_k; u_{k+1}, v)| \\
&\le |a(u_\infty; u_\infty, v) - a(u_k; u_{k+1}, v)| + |\langle \mathbf{R}(u_{k+1}), v\rangle| \\
&\le |a(u_\infty; u_\infty, v) - a(u_k; u_k, v)| \\
&\quad + |a(u_k; u_k - u_{k+1}, v)| + |\langle \mathbf{R}(u_{k+1}), v\rangle| \\
&\le \|Au_\infty - Au_k\|_{H^{-1}(\Omega)} \|\nabla v\|_\Omega \\
&\quad + C_a \|\nabla(u_k - u_{k+1})\|_\Omega \|\nabla v\|_\Omega + |\langle \mathbf{R}(u_k), v\rangle| \\
&\le C_A \|\nabla(u_\infty - u_k)\|_\Omega \|\nabla v\|_\Omega \\
&\quad + C_a \|\nabla(u_k - u_{k+1})\|_\Omega \|\nabla v\|_\Omega + |\langle \mathbf{R}(u_k), v\rangle|,
\end{aligned}$$

where we have used that A is Lipschitz (cf. (7)) and a is bounded (cf. (5)). Using Theorem 4.3, Lemma 4.2 and Theorem 4.7 it follows that the right-hand side in the last inequality tends to zero as k tends to infinity. □

As an immediate consequence of Theorems 4.3 and 4.8 we finally obtain the main result of this article.

Theorem 4.9 (Main result)**.** *Let $\{u_k\}_{k\in\mathbb{N}_0}$ denote the sequence of discrete solutions computed with the Adaptive Algorithm. If $\alpha(\cdot,\cdot)$ satisfies assumptions (3) and (17), then $\{u_k\}_{k\in\mathbb{N}_0}$ converges to the solution u of problem (2).*

We conclude this section with a couple of remarks.

Remark 4.10. The problem given by (1) is a particular case of the more general problem:

$$\begin{cases} -\nabla\cdot\left[\alpha(\,\cdot\,, |\nabla u|_{\mathcal{A}}^2)\mathcal{A}\nabla u\right] = f & \text{in } \Omega \\ \qquad\qquad\qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\alpha : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ and $f \in L^2(\Omega)$ satisfy the properties assumed in the previous sections, and $\mathcal{A} : \Omega \to \mathbb{R}^{d\times d}$ is symmetric for all $x \in \Omega$, and uniformly positive definite, i.e., there exist constants $\underline{a}, \overline{a} > 0$ such that

$$\underline{a}|\xi|^2 \leq \mathcal{A}(x)\xi \cdot \xi \leq \overline{a}|\xi|^2, \qquad \forall\, x \in \Omega,\, \xi \in \mathbb{R}^d.$$

If $\mathcal{A}$ is piecewise Lipschitz over an initial conforming mesh $\mathcal{T}_0$ of Ω, i.e., there exists $C_{\mathcal{A}} > 0$ such that

$$\|\mathcal{A}(x) - \mathcal{A}(y)\|_2 \leq C_{\mathcal{A}}|x - y|, \qquad \forall\, x, y \in T, \quad \forall\, T \in \mathcal{T}_0,$$

then the convergence results previously presented also hold for this problem.

Remark 4.11. We have assumed the use of *linear* finite elements for the discretization (see (9)). It is important to notice that the only place where we used this is in Proposition 3.3. The rest of the steps of the proof hold regardless of the degree of the finite element space. The use of linear finite elements is customary in nonlinear problems, because they greatly simplify the analysis. The numerical experiments of the next section show a competitive performance of the adaptive method for any tested polynomial degree (up to four).

5 Numerical Experiments

We conclude this article reporting on the behavior of the adaptive algorithm for some particular nonlinear problems. In the first subsection we study the convergence rate in terms of degrees of freedom for an exact solution and different functions $\alpha(\cdot,\cdot)$. In the second subsection we show the performance of the algorithm when approximating an unknown solution of a prescribed curvature equation.

5.1 Exact solution

Let us consider the problem

$$\begin{cases} -\nabla \cdot \left[\alpha(|\nabla u|^2)\nabla u\right] = f & \text{in}\, \Omega \\ u = g & \text{on}\, \partial\Omega, \end{cases} \tag{23}$$

where $\Omega \subset \mathbb{R}^2$ is the L-shaped domain given in Figure 1. In the following examples, in order to study experimentally the behavior of the Adaptive Algorithm, we consider (23) with different choices of the function α, defining f and g in each case so that the solution of the problem is the function u depicted in Figure 1, given in polar coordinates by

$$u(r, \varphi) = r^{\frac{2}{3}} \sin\left(\frac{2}{3}\varphi\right). \tag{24}$$

We consider the Adaptive Algorithm using different marking strategies, namely, *global refinement*, *maximum strategy* with $\theta = 0.7$ and *Dörfler's strategy* with $\theta = 0.5$ (see [11]).

We implemented the Adaptive Algorithm using the finite element toolbox ALBERTA [11]. We iterated the algorithm until the global error estimator was

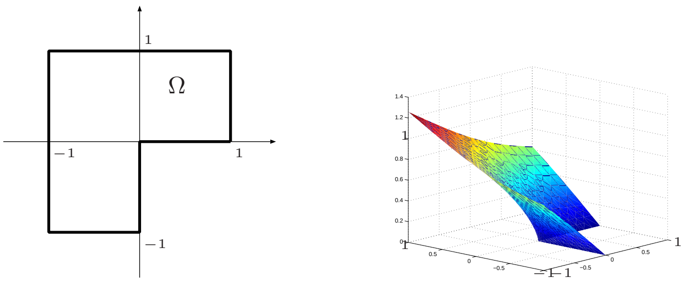

Figure 1: The domain Ω where the problem (23) is posed and the function u which is the solution of the problem in each example.

below 10^{-6} or the number of degrees of freedom exceeded 5×10^5. We tested the limits of our theory by trying with some functions α which did not satisfy all the assumptions of the theoretical results above.

Example 5.1 (Optimal rate of convergence when α satisfies the hypotheses). As a first example, in order to study experimentally the rate of convergence of the Adaptive Algorithm, we consider

$$\alpha(t) = \frac{1}{1+t} + \frac{1}{2}, \qquad t > 0,$$

which satisfies the hypotheses to guarantee the convergence (see Figure 2), i.e., α is a $\mathcal{C}^1$-function, and there exist positive constant c_a and C_a such that

$$c_a \leq \alpha(t^2) + 2t^2\alpha'(t^2) \leq C_a, \qquad \forall\, t > 0, \tag{25}$$

and

$$\alpha \text{ is monotone decreasing, i.e., } \alpha'(t) \leq 0 \text{ for all } t > 0. \tag{26}$$

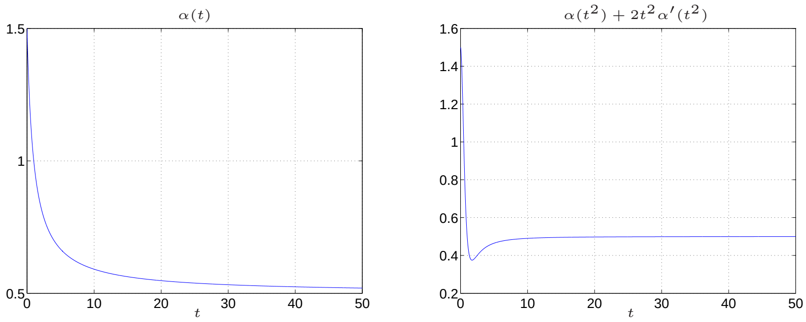

Figure 2: The function $\alpha(t) = \frac{1}{1+t} + \frac{1}{2}$, of Example 5.1 satisfies the properties we require to guarantee the convergence.

In Figure 3 we plot the $H^1(\Omega)$-error versus the number of degrees of freedom (DOFs), for finite elements of degree $\ell = 1, 2, 3, 4$. In this case, the rate of the convergence is optimal for adaptive strategies, that is, $\|u - u_k\|_{H^1(\Omega)} =$

$O(\mathrm{DOFs}_k^{-\ell/2})$. For global refinement, the observed order of convergence is $\mathrm{DOFs}_k^{-1/3}$ for all tested polynomial degrees, due to the fact that the solution u belongs to $H^{1+\delta}(\Omega)$, for all $0 < \delta < \frac{2}{3}$, and does not belong to $H^{1+2/3}(\Omega)$.

Note that, although the theory only guarantees the plain convergence for linear elements (cf. Theorem 4.9), the numerical results suggest that the method works for any polynomial degree (see Remark 4.11), and the convergence rate is optimal.

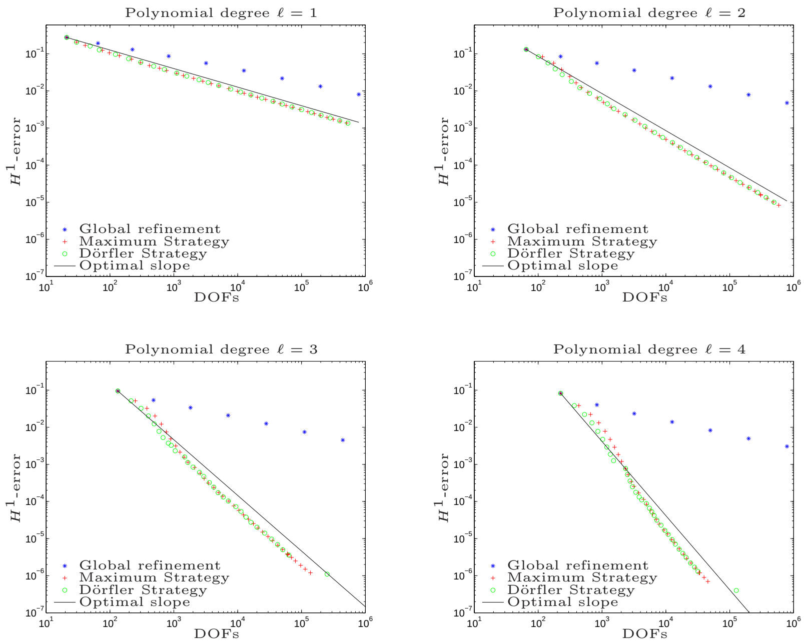

Figure 3: Error versus DOFs for Example 5.1. We present the $H^1(\Omega)$-error between the exact solution and discrete solutions, versus the number of degrees of freedom (DOFs) used to represent each of them. We note that the convergence rate is optimal for the considered adaptive strategies, but not for global refinement, due to the fact that the solution u is not sufficiently smooth. In this case, $\alpha(t) = \frac{1}{1+t} + \frac{1}{2}$ satisfies all the properties established to guarantee the convergence with linear finite elements. The numerical experiments suggest that the method converges with optimal rate for any polynomial degree.

Example 5.2 (About the hypothesis (25)). We consider the function

$$\alpha(t) = \frac{1}{1+t} + \frac{1}{10}, \qquad t > 0,$$

which is monotone decreasing, i.e., satisfies (26), but not (25), as it is shown in Figure 4. Since (25) guarantees the well-posedness of problem (23) (uniqueness and stability), we could be facing an example with multiple solutions.

In Figure 5 we plot the $H^1(\Omega)$-error versus the number degrees of freedom, for different polynomial degrees. For $\ell = 3$ and $\ell = 4$ the algorithm stopped with an estimator below the desired tolerance 10^{-6}, although the error is around

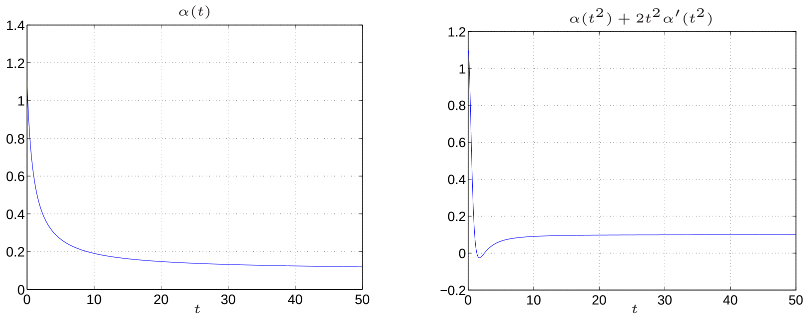

Figure 4: The function $\alpha(t) = \frac{1}{1+t} + \frac{1}{10}$ of Example 5.2 does not satisfy $\alpha(t^2) + 2t^2\alpha'(t^2) > 0$ for all $t > 0$.

10^{-2} in all cases. On the other hand, as we can see in Figure 6, the global error estimator decreases with optimal rate for the adaptive strategies, indicating that the adaptive algorithm may be converging to another solution of the nonlinear problem, different from the one given by (24).

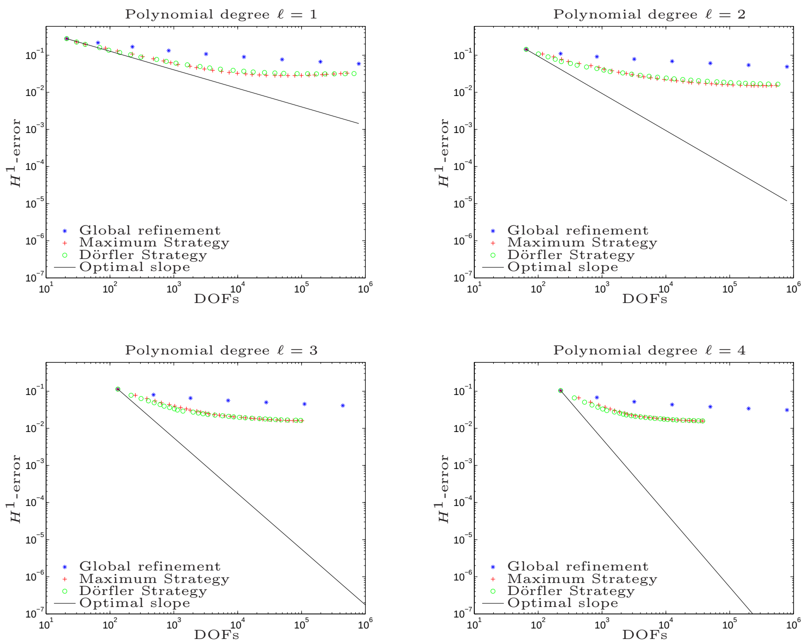

Figure 5: Error versus DOFs for Example 5.2. We present the $H^1(\Omega)$-error between the exact and discrete solutions, versus DOFs. We observe that the method does not converge, but the error stagnates around 10^{-2}. Since $\alpha(t) = \frac{1}{1+t} + \frac{1}{10}$ does not satisfy the condition which guarantees uniqueness of solutions, and based on the fact that the a posteriori error estimator does tend to zero (see Figure 6) we conclude that the method converges to *another solution* of the same problem, different from the one given by (24).

Based on this remark, it seems that the adaptive algorithm converges to a solution u_1 such that $\|u - u_1\|_{H^1(\Omega)} \approx 10^{-2}$. We recall that α does not satisfy the condition (25) which guarantees the uniqueness of solutions.

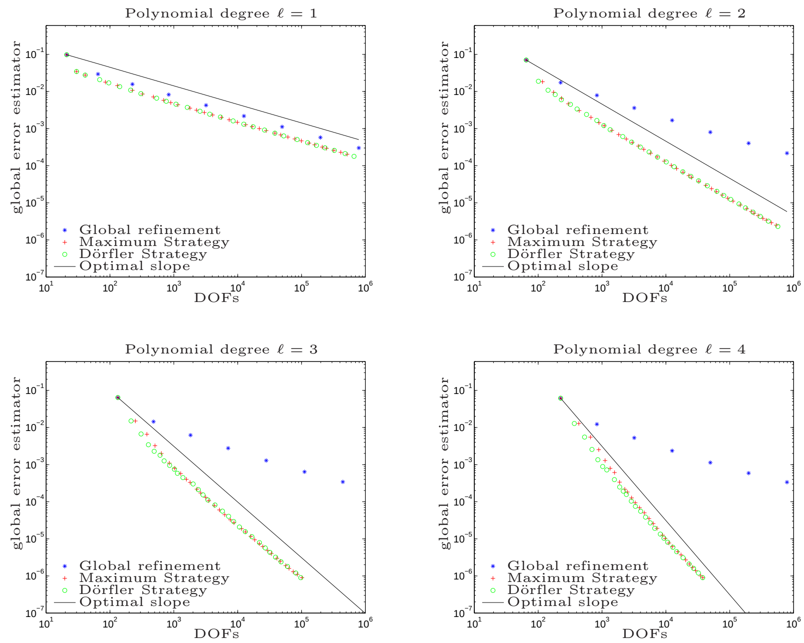

Figure 6: Estimator versus DOFs for Example 5.2. We present the global error estimator $\eta_k(\mathcal{T}_k)$ versus DOFs used to represent each discrete solution. We note that for the adaptive strategies the global error estimator decreases with the optimal rate for the H^1-error, although the error does not tend to zero (see Figure 5). In this case, $\alpha(t) = \frac{1}{1+t} + \frac{1}{10}$ does not satisfy the condition which guarantees uniqueness of solutions. It seems that the method converges to *another solution* of the problem.

Example 5.3 (About the hypothesis (26)). We consider

$$\alpha(t) = -\frac{1}{2}\,\mathrm{e}^{-\frac{3}{2}t} + 1, \qquad t > 0,$$

which satisfies the hypothesis (25) related to the well-posedness of the problem but not (26), because α is monotone increasing (see Figure 7).

In Figure 8 we plot $H^1(\Omega)$-error versus the number of degrees of freedom, for finite elements of degrees $\ell = 1, 2, 3, 4$. Note that in this case the optimal convergence rate $\mathrm{DOFs}^{-\ell/2}$ is still observed for the adaptive algorithm. This is an indication that the assumption (26) about α being monotone decreasing can be superfluous, and only an artificial requirement for the presented proof (see Lemma 4.1). A more detailed study about this hypothesis is subject of future research, and beyond the scope of this article.

Example 5.4. Finally, we consider an extreme case, with

$$\alpha(t) = 2 - \frac{\sqrt{t}}{1 + \sqrt{t}}, \qquad t > 0,$$

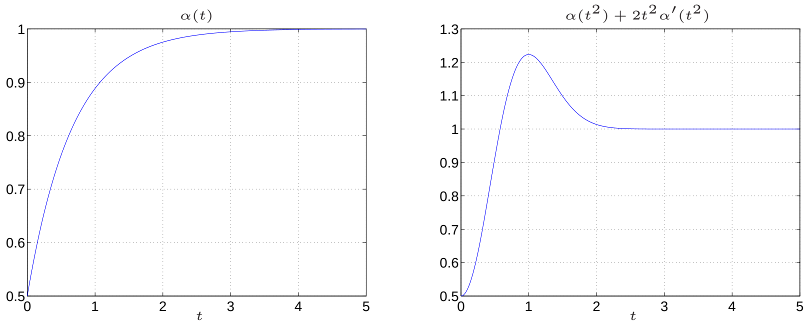

Figure 7: The function $\alpha(t) = -\frac{1}{2}\,\mathrm{e}^{-\frac{3}{2}t} + 1$ of Example 5.3 satisfies the hypothesis (25) but is monotone increasing.

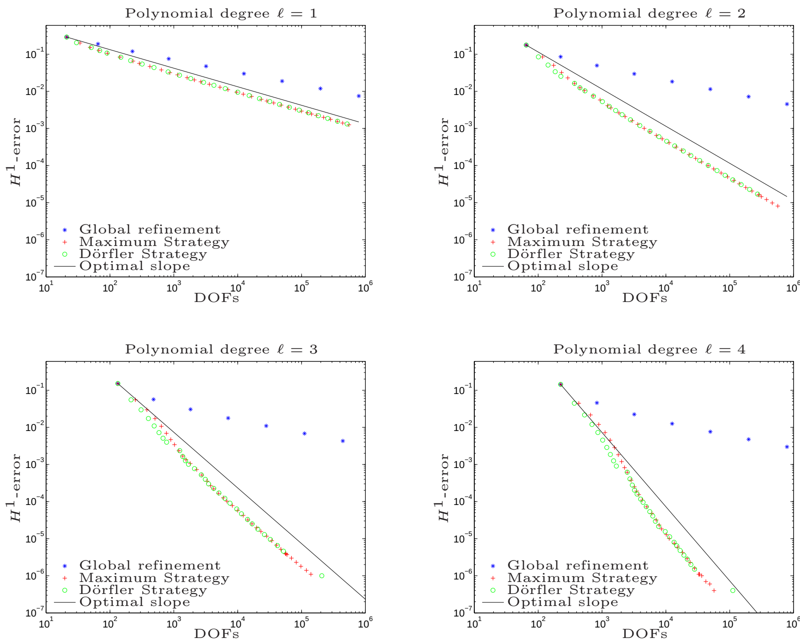

Figure 8: Error versus DOFs for Example 5.3. We present the $H^1(\Omega)$-error between the exact and discrete solutions, versus DOFs. We note that the convergence rate is optimal for the considered adaptive strategies, although the function $\alpha(t) = -\frac{1}{2}\,\mathrm{e}^{-\frac{3}{2}t} + 1$ is not monotone decreasing. This could mean that this hypothesis is not necessary for the convergence of the Adaptive Algorithm, which performs better than expected by our theory.

which satisfies (25) and (26), but $\lim_{t\to 0^+} \alpha'(t) = -\infty$, as can be observed in Figure 9. This means that α is not Lipschitz continuous. Since we only require that $|t\alpha'(t)|$ is bounded (cf. (25)), it still satisfies the assumptions of the convergence theory, and optimality is observed regardless of the fact that $\lim_{t\to 0^+} \alpha'(t) = -\infty$.

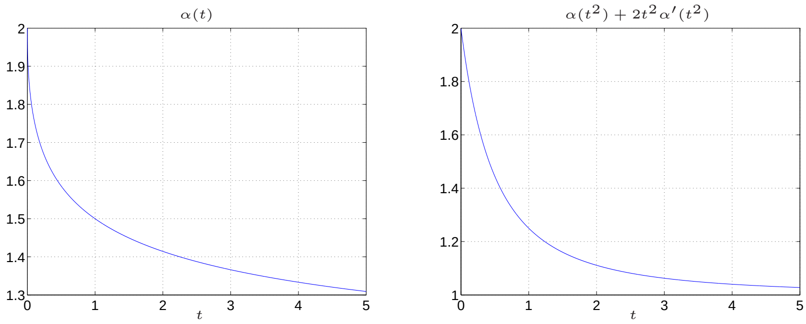

Figure 9: The function $\alpha(t) = 2 - \frac{\sqrt{t}}{1+\sqrt{t}}$ of Example 5.4 satisfies our assumptions which do not require that α is Lipschitz continuous.

Figure 10 shows $H^1(\Omega)$-error versus DOFs, for polynomial degrees $\ell = 1, 2, 3, 4$. We obtain optimal convergence rate DOFs$^{-\ell/2}$ for the adaptive strategies. Thus, even when α is not Lipschitz, the convergence rate is optimal. As a consequence we conclude that it should not be necessary to make additional assumptions in order to prove optimality.

5.2 Unknown solution

In this section we use the Adaptive Algorithm to approximate a solution to a prescribed mean curvature problem. We consider the problem

$$\begin{cases} -\nabla \cdot \left[\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right] = f & \text{in } \Omega = (-1,1)\times(-1,1) \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{27}$$

with

$$f(x) = \begin{cases} 5 & \text{if } |x| \le 1/3, \\ -3 & \text{if } 1/3 < |x| \le 2/3, \\ 0 & \text{otherwise.} \end{cases}$$

The function $\alpha(t) = 1/\sqrt{1+t}$ corresponding to this equation does not fulfill (25) because $\alpha(t), \alpha'(t) \to 0$ as $t \to \infty$. Nevertheless, for many right-hand side functions f, as the one stated above, the solution belongs to $W^{1,\infty}(\Omega)$. This implies that $|\nabla u|$ is bounded and α could be replaced by a function satisfying (25) without changing the solution. This is not needed in practice, but is rather a theoretical tool for proving that the assumptions hold in some sense.

We experimented with several right-hand sides f and observed that the method performs robustly whenever $u \in W^{1,\infty}(\Omega)$. When the solution has an unbounded gradient, the method produces a sequence with a maximum value

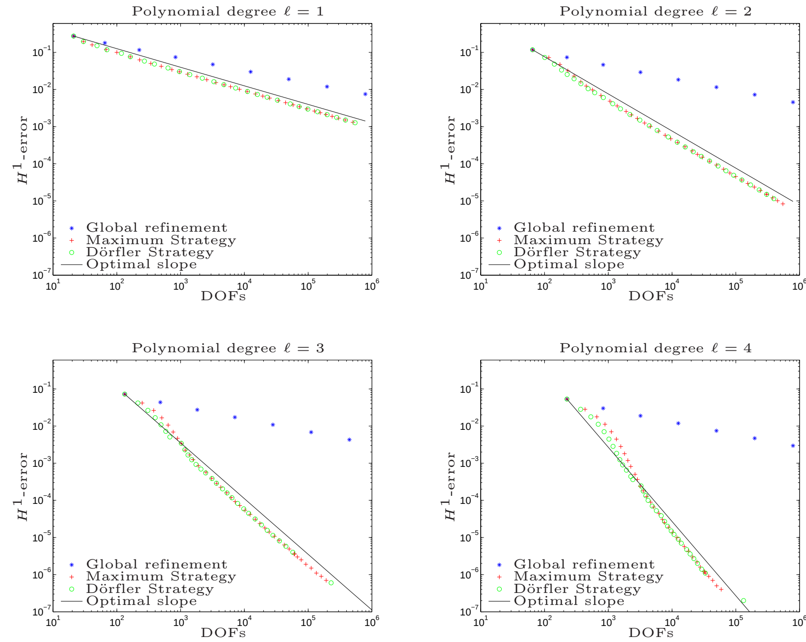

Figure 10: Error versus DOFs for Example 5.4. We present the $H^1(\Omega)$-error between the exact and discrete solutions, versus DOFs. We note that the convergence rate is optimal for the considered adaptive strategies, even though the function $\alpha(t) = 2 - \frac{\sqrt{t}}{1+\sqrt{t}}$ has an infinite derivative at 0. This example falls inside the theory, and the fact that α is not Lipschitz continuous does not destroy the optimality of the sequence of discrete solutions.

increasing to infinity. This is a drawback of our method, since it cannot be used to approximate singular solutions (not belonging to $W^{1,\infty}(\Omega)$), if α does not satisfy (25). It is worth recalling that adaptivity is still a good tool for approximating regular solutions, specially taking into account the fact that in this context *regularity* is a concept relative to the polynomial degree. Using adaptivity with higher order finite elements can improve drastically the performance even when the solution is in $H^2(\Omega) \cap W^{1,\infty}(\Omega)$.

We present in Figure 11 a picture of the solution obtained with our method and several meshes at different iteration steps, for polynomial degrees 1, 2 and 3. It is worth observing the stronger grading obtained for higher polynomial degrees. This obeys the fact that we mention above, that the singularity of the solution is relative to the polynomial degree of the finite element space. Recall that in practice adaptive methods with polynomials of degree ℓ on domains in $\mathbb{R}^d$ converge with order $\mathrm{DOF}^{-\ell/d}$. In order to obtain this rate with uniform meshes, we would need the solution to belong to $H^{\ell+1}(\Omega)$.

References

[1] E. Bänsch, P. Morin, R. H. Nochetto, *An adaptive Uzawa FEM for the Stokes Problem. Convergence without the inf-sup condition*, SIAM J. Numer. Anal. 40 (2002), 1207–1229.

[2] W. Dörfler, *A convergent adaptive algorithm for Poisson's equation*, SIAM J. Numer. Anal. 33 (1996), 1106–1124.

[3] E. M. Garau, P. Morin, and C. Zuppa, *Convergence of adaptive finite element methods for eigenvalue problems*, Math. Models Methods Appl. Sci. **19** (2009), no. 5, 721–747.

[4] E. M. Garau and P. Morin, *Convergence and quasi-optimality of adaptive FEM for Steklov eigenvalue problems*, to appear in IMA J. Numer. Anal. (2010)

[5] E. M. Garau, P. Morin, and C. Zuppa, *Optimal complexity of AFEM for quasi-linear problems*, in preparation (2010).

[6] W. Han, S. Jensen, I. Shimansky, *The Kačanov method for some nonlinear problems*, Applied Numerical Mathematics **24** (1997), 57–79,

[7] Y. Kondratyuk, R. Stevenson *An optimal adaptive finite element method for the Stokes problem*, SIAM J. Numer. Anal. 46 (2008), 747–775.

[8] L. Diening, C. Kreuzer, *Linear convergence of an adaptive finite element method for the p-Laplacian equation*, SIAM J. Numer. Anal. 46 (2008), 614–638.

[9] P. Morin, K. G. Siebert, and A. Veeser, *A basic convergence result for conforming adaptive finite elements*, Math. Models Methods Appl. Sci. **18** (2008), no. 5, 707–737.

[10] R.H. Nochetto, K. G. Siebert, A. Veeser, *Theory of Adaptive Finite Element Methods: An Introduction*, In Multiscale, Nonlinear and Adaptive

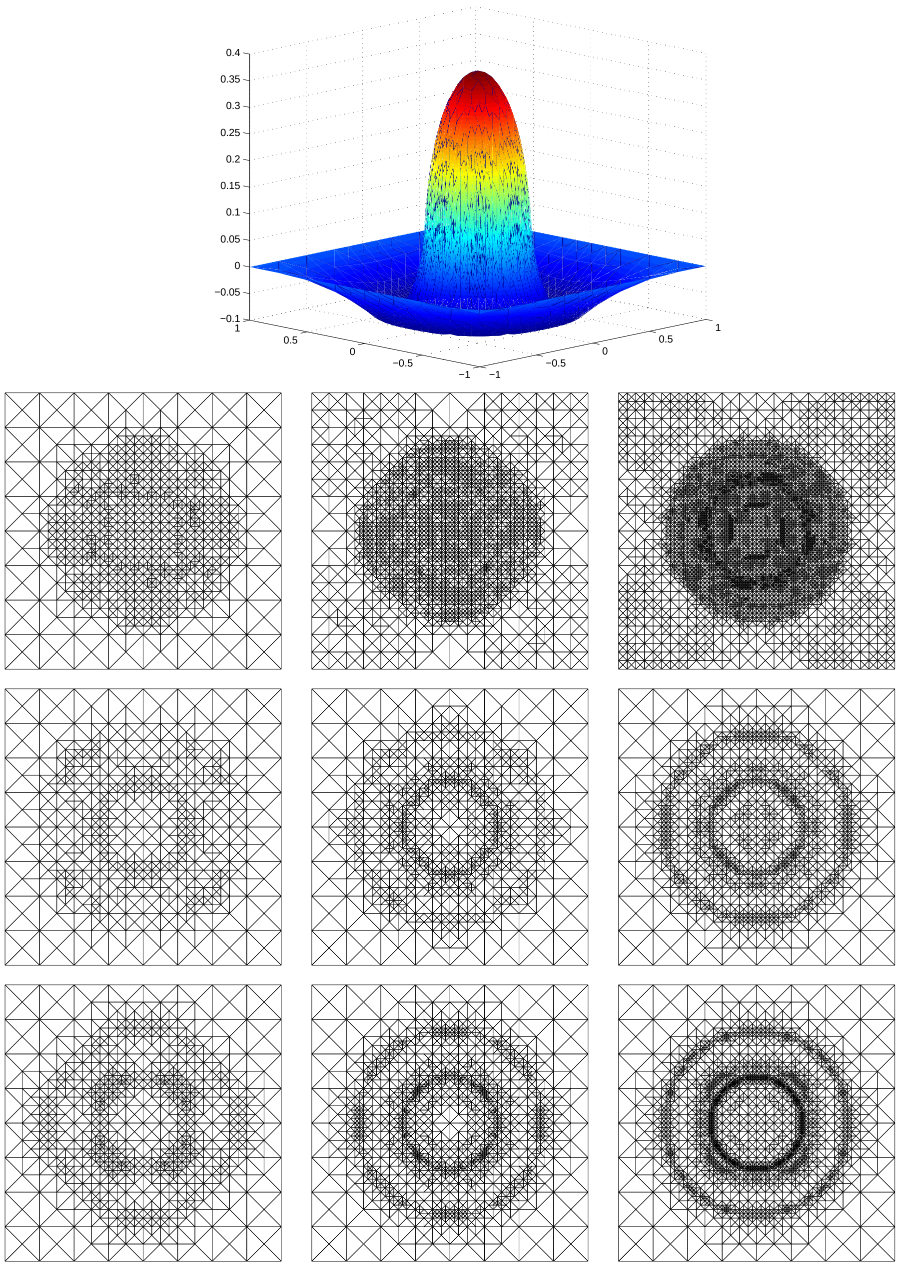

Figure 11: Adaptive meshes obtained when solving the prescribed mean curvature equation (27) with polynomials of degree 1 (top), 2 (middle), 3 (bottom). The meshes correspond to iteration count 10 (left), 15 (middle) and 20 (right). It is worth observing the higher grading presented by the meshes corresponding to higher polynomial degree.

Approximation: Dedicated to Wolfgang Dahmen on the Occasion of his 60th Birthday, R. DeVore, A. Kunoth, eds., Springer, 2009.

[11] A. Schmidt and K. G. Siebert, *Design of adaptive finite element software*, Lecture Notes in Computational Science and Engineering, vol. 42, Springer-Verlag, Berlin, 2005, The finite element toolbox ALBERTA, With 1 CD-ROM (Unix/Linux).

[12] K. G. Siebert, *A convergence proof for adaptive finite elements without lower bound*, IMA J. Numer. Anal., in press (2010).

[13] R. Stevenson, *The completion of locally refined simplicial partitions created by bisection*, Math. Comp. **77** (2008), no. 261, 227–241 (electronic).

[14] E. Zarantonello, *Solving functional equations by contractive averaging*, Mathematics Research Center Report #160 (1960), Madison, WI.

[15] E. Zeidler, *Nonlinear functional analysis and its applications. II/B*, Springer-Verlag, New York, (1990), Nonlinear monotone operators, Translated from the German by the author and Leo F. Boron.

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