Chapter 4. Vibrating Strings and Membranes

Section 4.4

4.4.1 (a) Natural frequencies are $c\sqrt{\lambda}$, but $\lambda = (n\pi/L)^2$. Thus frequencies are $n\pi c/L, n = 1, 2, 3, \ldots$

4.4.1 (b) Natural frequencies are $c\sqrt{\lambda}$. The boundary condition $\phi(0) = 0$ implies $\phi = c_1 \sin\sqrt{\lambda}x$, while $d\phi/dx(H) = 0$ yields $\sqrt{\lambda}H = (m - \frac{1}{2})\pi$ with $m = 1, 2, 3$. Thus the frequencies are $(m - \frac{1}{2})\pi c/H$ and the eigenfunctions are $\sin(m - \frac{1}{2})\pi x/H$.

4.4.2 (c) By separation of variables, $u = \phi(x)h(t)$, $\frac{d^2h}{dt^2} = -\lambda h$ and $T_0\frac{d^2\phi}{dx^2} + (\alpha + \lambda\rho_0)\phi = 0$. With $\phi(0) = 0$ and $\phi(L) = 0, (\alpha + \lambda\rho_0)/T_0 = (n\pi/L)^2, n = 1, 2, 3, \ldots$ and $\phi = \sin n\pi x/L$. In general $h(t)$ involves a linear combination of $\sin\sqrt{\lambda}t$ and $\cos\sqrt{\lambda}t$, but the homogeneous initial condition $u(x, 0) = 0$ implies there are no cosines. Thus by superposition

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\sqrt{\lambda_n}t \sin n\pi x/L,$$

where the frequencies of vibration are $\sqrt{\lambda_n} = \sqrt{\frac{(n\pi/L)^2T_0 - \alpha}{\rho_0}}$. The other initial condition, $f(x) = \sum_{n=1}^{\infty} A_n\sqrt{\lambda_n}\sin n\pi x/L$, determines A_n

$$A_n\sqrt{\lambda_n} = \frac{2}{L}\int_0^L f(x)\sin n\pi x/L\, dx.$$

4.4.3 (b) By separation of variables, $u = \phi(x)h(t)$, $\frac{\rho_0 h'' + \beta h'}{hT_0} = \frac{\phi''}{\phi} = -\lambda$. The boundary conditions $\phi(0) = 0$ and $\phi(L) = 0$ yield $\lambda = (n\pi/L)^2$ with $\phi = \sin n\pi x/L$, $n = 1, 2, 3, \ldots$. The time-dependent equation has constant coefficients,

$$\rho_0 h'' + \beta h' + \left(\frac{n\pi}{L}\right)^2 T_0 h = 0,$$

and hence can be solved by substitution $h = e^{rt}$. This yields the quadratic equation

$$\rho_0 r^2 + \beta r + \left(\frac{n\pi}{L}\right)^2 T_0 = 0,$$

whose roots are

$$r = \frac{-\beta \pm \sqrt{\beta^2 - 4\rho_0 T_0 (n\pi/L)^2}}{2\rho_0}.$$

Since $\beta^2 < 4\rho_0 T_0(\pi/L)^2$, the discriminant is < 0 for all n:

$$r = -\frac{\beta}{2\rho_0} + iw_n, \text{ where } w_n = \sqrt{\frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2 - \frac{\beta^2}{4\rho_0^2}}.$$

Real solutions are $h = e^{-\beta t/2\rho_0}(\sin w_n t, \cos w_n t)$. Thus by superposition

$$u = e^{-\beta t/2\rho_0}\sum_{n=1}^{\infty}(a_n \cos w_n t + b_n \sin w_n t)\sin\frac{n\pi x}{L}.$$

The initial condition $u(x, 0) = f(x)$ determines a_n, $a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$, while $\frac{\partial u}{\partial t}(x, 0) = g(x)$ is a little more complicated, $g(x) = \sum_{n=1}^{\infty} b_n w_n \sin\frac{n\pi x}{L} - \frac{\beta}{2\rho_0}\underbrace{\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}}_{f(x)}$, and thus

$$b_n w_n = \frac{\beta a_n}{2\rho_0} + \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx.$$