Convergent Spectral Approximations for the Thermomechanical Processes in Shape Memory Alloys

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Abstract: In this article discrete spectral approximations to the nonlinear evolutionary partial differential equations that model the dynamics of thermomechanical solid-solid phase transitions in one-dimensional shape memory alloys with non-convex Landau-Ginzburg potentials are constructed. By using the theories of analytic semigroups and interpolation spaces and a generalization of Gronwall's lemma for singular kernels, the convergence of the approximations is proved. For the alloy $Au_{23}Cu_{30}Zn_{47}$ numerical results are shown under different external distributed actions and initial conditions.

Keywords: Shape Memory Alloys, non-convex potential, hysteresis, conservation laws, initial-boundary value problem, spectral approximations.

AMS Subject Classifications: 35A35, 35A40, 35M05, 73U05, 73C35.

1. Introduction

In this article we consider the following one-dimensional nonlinear initial-boundary value problem:

$$\rho u_{tt} - \beta\rho u_{xxt} + \gamma u_{xxxx} = f(x,t) + \left(2\alpha_2(\theta-\theta_1)u_x - 4\alpha_4 u_x^3 + 6\alpha_6 u_x^5\right)_x, \quad x \in (0,1),\ 0 \le t \le T \tag{1.1a}$$

$$C_v\theta_t - k\theta_{xx} = g(x,t) + 2\alpha_2\theta u_x u_{xt} + \beta\rho u_{xt}^2, \quad x \in (0,1),\ 0 \le t \le T \tag{1.1b}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = v_0(x),\ \theta(x,0) = \theta_0(x), \quad x \in (0,1) \tag{1.1c}$$

$$u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \quad 0 \le t \le T \tag{1.1d}$$

$$\theta_x(0,t) = \theta_x(1,t) = 0, \quad 0 \le t \le T \tag{1.1e}$$

System (1.1a-e) arises from the conservation laws governing the thermomechanical processes in one-dimensional Shape Memory Alloys (SMA). These processes are characterized by solid-solid phase

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transitions (martensitic transformations). Equations (1.1a) and (1.1b) reflect the conservation of linear momentum and energy, respectively. The functions and variables present in system (1.1a-e) have the following physical meaning: $u(x,t)$ = transverse displacement, $\theta(x,t)$ = absolute temperature, C_v = specific heat, k = thermal conductivity coefficient, β = viscosity constant, $f(x,t)$ = distributed loads (input), $g(x,t)$ = distributed heat sources (input), T = prescribed final time, α_2, α_4, α_6, θ_1, γ are positive constants -depending on the material being considered- appearing in the free energy potential which is taken in the Landau-Ginzburg form

$$\Psi(\epsilon,\epsilon_x,\theta) = -C_v\theta\ln\left(\frac{\theta}{\theta_2}\right) + C_v\theta + C + \alpha_2(\theta-\theta_1)\epsilon^2 - \alpha_4\epsilon^4 + \alpha_6\epsilon^6 + \frac{\gamma}{2}\epsilon_x^2 \tag{1.2}$$

where $\epsilon = u_x$ is the linearized shear strain. The constants θ_1 and θ_2 in (1.2) are two critical temperatures and C represents a fixed energy reference level. The body is assumed to be a simply supported unit-length beam thermally insulated at both ends.

The PDE's in (1.1a-b) are coupled and nonlinear due to the terms coming from the partial derivatives of the free energy. The first equation can be regarded as a nonlinear hyperbolic equation in u while the second is a nonlinear parabolic equation in θ (for a detailed derivation of equations (1.1a-b) see [31]).

Although there are several representations for the free energy potential of pseudoelastic materials (see for instance [13], [14], [29], [30], [34]) the form (1.2) seems to be the simplest one which is able to reproduce several phenomena -such as hysteresis, shape memory and superelasticity- observed in real materials under different external thermomechanical actions. For values of θ close to θ_1, Ψ is a nonconvex function of ϵ and the stress-strain laws obtained from (1.2) are strongly temperature-dependent (see Figure 1). At low temperatures these curves exhibit an elasto-plastic behavior at small loads and a second elastic branch at large loads, which permits the body to withstand forces beyond the plastic yield, after which, subsequent unloading produces residual deformation. In the intermediate temperature range the behavior is superelastic, also called pseudoelastic. Here, a plastic yield is also found. However, loading beyond this plastic yield followed by complete unloading does not lead to residual deformation because of the existence of an intermediate elastic branch which the body reaches by creeping back after the load falls beyond a certain critical value. Finally, in the high temperature range the behavior is almost linearly elastic with higher modulus of elasticity for higher temperatures. Hysteresis loops are observed in the stress-strain curves at low and intermediate temperatures (see [31] and the references therein).

Due to their unique characteristics SMA have already found a broad spectrum of applications among which we find orthodontic and other dental devices ([4]), heat engines, temperature switches and fuses, pipe coupling devices ([16]), hybrid composites ([27]) and several interesting applications in Medicine ([10], [16], [28]).

Since the discovery of NiTinol (a Nikel-Titanium alloy) by Buehler ([20]) in 1962 several mathematical models were proposed and studied ([1], [2], [3], [13], [14], [19], [21], [22], [23], [35]). Most of this models, however, were static and did not take into account the strong coupling between the mechanical and thermal properties, which is one of the distinguishing features possessed by SMA. It was not until recent years that mathematical models were able to deal with most of the unusual properties of SMA and, at the same time, to allow for the inclusion of boundary and distributed external actions that can be used as control variables ([24], [25], [29], [30], [33], [34], [31]). An extensive account on the recent advances in the mathematical modelling of SMA can be found in [6]. This article follows the approach introduced in [31].

2. State-Space Formulation and Preliminaries

In this section we shall formulate the initial-boundary value problem (1.1a-e) as an abstract semilinear Cauchy problem in an appropriate Hilbert space and briefly recall some preliminaries which will be needed later on.

We define the admissible parameter set $\mathcal{Q} \doteq \{q = (\rho, k, C_v, \beta, \alpha_2, \alpha_4, \alpha_6, \theta_1, \gamma) | q \in \mathbb{R}^9_+\}$, and for $q \in \mathcal{Q}$ the state space Z_q as the Hilbert space $H^1_0(0,1) \cap H^2(0,1) \times L^2(0,1) \times L^2(0,1)$ with the inner product

$$\left\langle \begin{pmatrix} u \\ v \\ \theta \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\theta} \end{pmatrix} \right\rangle_q \doteq \gamma \int_0^1 u''(x)\overline{\tilde{u}''(x)}\,dx + \rho\int_0^1 v(x)\overline{\tilde{v}(x)}\,dx + \frac{C_v}{k}\int_0^1 \theta(x)\overline{\tilde{\theta}(x)}\,dx.$$

Next, for $q \in \mathcal{Q}$, the operator A_q on Z_q is defined by

$$D(A_q) = \left\{ \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} \in Z_q \;\middle|\; \begin{array}{l} u \in H^4(0,1),\ u(0) = u(1) = u''(0) = u''(1) = 0 \\ v \in H_0^1(0,1) \cap H^2(0,1) \\ \theta \in H^2(0,1), \theta'(0) = \theta'(1) = 0 \end{array} \right\}$$

and for $z = \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} \in D(A_q)$,

$$A_q \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} \doteq \begin{pmatrix} 0 & I & 0 \\ -\frac{\gamma}{\rho} D^4 & \beta D^2 & 0 \\ 0 & 0 & \frac{k}{C_v} D^2 \end{pmatrix} \begin{pmatrix} u \\ v \\ \theta \end{pmatrix}$$

where $D^n \doteq \frac{\partial^n}{\partial x^n}$.

We assume that the functions $f(x,t)$, $g(x,t)$ satisfy the following hypothesis.

(H1). *For each fixed $t \geq 0$, the functions $f(x,t)$, $g(x,t)$ are in $L^2(0,1)$ and there exist nonnegative functions $K_f(x)$, $K_g(x) \in L^2(0,1)$ such that*

$$|f(x,t_1) - f(x,t_2)| \leq K_f(x)|t_1 - t_2|, \quad |g(x,t_1) - g(x,t_2)| \leq K_g(x)|t_1 - t_2|$$

for all $x \in (0,1)$, $t_1, t_2 \in [0,T]$.

We also define $z_0(x) = \begin{pmatrix} u_0(x) \\ v_0(x) \\ \theta_0(x) \end{pmatrix}$ and $F(q,t,z) : \mathcal{Q} \times \mathbb{R}_0^+ \times Z_q \to Z_q$ by

$$F(q,t,z) = \begin{pmatrix} 0 \\ f_2(q,t,z) \\ f_3(q,t,z) \end{pmatrix},$$

where

$$\rho f_2(q,t,z)(x) = f(x,t) + \left(2\alpha_2(\theta - \theta_1)u_x - 4\alpha_4 u_x^3 + 6\alpha_6 u_x^5\right)_x,$$
$$C_v f_3(q,t,z)(x) = g(x,t) + 2\alpha_2 \theta u_x u_{xt} + \beta\rho u_{xt}^2.$$

With the above notation, the IBVP (1.1a-e) can be formally written as the following semilinear Cauchy problem in the Hilbert space Z_q:

$$(\mathcal{P}) \begin{cases} \frac{d}{dt} z(t) = A_q z(t) + F(q,t,z), \quad 0 \leq t \leq T \\ z(0) = z_0 \end{cases} \tag{2.1}$$

The following results can be easily derived from theorems 3.7 and 3.11 in [31] with only slight modifications in order to take into account for the slightly different boundary conditions being considered here. Since the modifications needed are trivial and not important for the goals pursued by this article, we do not give details here.

Theorem 2.1. ([31]) *Let $q \in \mathcal{Q}$, $A_q : D(A_q) \subset Z_q \to Z_q$ as previously defined. Then*

i) *A_q is dissipative;*

ii) *The adjoint A_q^* of A_q is also dissipative and is given by $D(A_q^*) = D(A_q)$,*

$$A_q \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} \doteq \begin{pmatrix} 0 & -I & 0 \\ \frac{\gamma}{\rho} D^4 & \beta D^2 & 0 \\ 0 & 0 & \frac{k}{C_v} D^2 \end{pmatrix} \begin{pmatrix} u \\ v \\ \theta \end{pmatrix}$$

iii) *The operator A_q has pure point spectrum $\sigma_p(A_q)$ given by*

$$\sigma_p(A_q) = \{\lambda_n^+\}_{n=1}^\infty \cup \{\lambda_n^-\}_{n=1}^\infty \cup \{\alpha_n\}_{n=0}^\infty,$$

where

$$\lambda_n^{+,-} = \sqrt{\mu_n}\left(-r(q) \pm \sqrt{r^2(q)-1}\right), \quad \alpha_n = -\frac{k}{C_v}n^2\pi^2$$

and

$$\mu_n = \frac{\gamma n^4\pi^4}{\rho}, \quad r(q) = \frac{\beta\sqrt{\rho}}{2\sqrt{\gamma}}.$$

The corresponding normalized eigenvectors in Z_q are, respectively,

$$\begin{pmatrix} e_n(x) \\ \lambda_n^+ e_n(x) \\ 0 \end{pmatrix}_{n=1,2,\cdots}, \quad \begin{pmatrix} k_n e_n(x) \\ k_n\lambda_n^- e_n(x) \\ 0 \end{pmatrix}_{n=1,2,\cdots}, \quad \begin{pmatrix} 0 \\ 0 \\ \chi_n(x) \end{pmatrix}_{n=0,1,\cdots},$$

where

$$k_n^2 = \frac{\mu_n + |\lambda_n^+|^2}{\mu_n + |\lambda_n^-|^2}, \quad e_n(x) = \left[\frac{2}{\rho(\mu_n + |\lambda_n^+|^2)}\right]^{1/2}\sin(\pi n x), \quad n = 1,2,\cdots,$$

$$\chi_0(x) = \left(\frac{k}{C_v}\right)^{1/2}, \qquad \chi_n(x) = \left(\frac{2k}{C_v}\right)^{1/2}\cos(\pi n x), \quad n = 1,2,\cdots;$$

iv) *The operator A_q generates an analytic semigroup of contractions $T_q(t)$ on Z_q.*

Theorem 2.2. ([31]) *(Local existence of solutions) Let $q \in \mathcal{Q}$ and A_q as defined above. Then for any initial data $z_0 \in D(A_q)$ there exists $t_1 = t_1(z_0)$ such that the IVP $(\mathcal{P})$ has a unique classical solution $z_q(t) \in C([0,t_1) : Z_q) \cap C^1((0,t_1) : Z_q)$.*

It will be useful to introduce some notation for certain interpolation spaces. If X is a Banach space and $p \geq 1$, $L_*^p(X)$ will denote the Banach space of all Bochner measurable mappings $u : [0,\infty) \to X$ such that $\|u\|_{L_*^p(X)}^p \doteq \int_0^\infty \|u(t)\|_X^p \frac{dt}{t} < \infty$. If X_0, X_1 are two Banach spaces with X_0 continuously and densely embedded in X_1, $p \geq 1$ and $\nu \in (0,1)$, we denote with $(X_0, X_1)_{\nu,p}$ the space of averages, -or "real" interpolation space-

$$(X_0,X_1)_{\nu,p} \doteq \left\{ x \in X_1 \;\middle|\; \begin{array}{l} \exists u_i : [0,\infty) \to X_i, i = 0,1, \quad t^{-\nu}u_0 \in L_*^p(X_0), \\ t^{1-\nu}u_1 \in L_*^p(X_1) \text{ and } x = u_0(t) + u_1(t) \text{ a.e.} \end{array} \right\}.$$

Endowed with the norm

$$\|x\|_{(X_0,X_1)_{\nu,p}} \doteq \inf\left\{ \|t^{-\nu}u_0\|_{L_*^p(X_0)} + \|t^{1-\nu}u_1\|_{L_*^p(X_1)} \;\middle|\; \begin{array}{l} t^{-\nu}u_0 \in L_*^p(X_0), \\ t^{1-\nu}u_1 \in L_*^p(X_1) \text{ and} \\ x = u_0(t) + u_1(t) \text{ a.e.} \end{array} \right\},$$

$(X_0,X_1)_{\nu,p}$ is a Banach space. In the particular case when $p = 2$ and X_0, X_1 are Hilbert spaces, we shall denote $(X_0,X_1)_{\nu,2} = [X_0,X_1]_\nu$ (see [5]).

If B is the infinitesimal generator of an analytic semigroup $S(t)$ on a Banach space X such that 0 belongs to the resolvent set of B, $\rho(B)$, then the fractional δ-powers $(-B)^\delta$ are well defined, closed, linear, invertible operators for any $\delta > 0$ (see [26, pp. 69-75]). Moreover, $D\left((-B)^\delta\right)$ endowed with the topology of the graph norm $\|x\|_\delta \doteq \left\|(-B)^\delta x\right\|$ is a Banach space. The following result can be found in [5].

Theorem 2.3. ([5]) *Let X be a Hilbert space and B the infinitesimal generator of an analytic semigroup on X such that $0 \in \rho(B)$. Then, for any $\delta \in (0,1)$, the Hilbert Space $\left(D\left((-B)^\delta\right), \|\cdot\|_\delta\right)$ is isomorphic to the interpolation space $[D(B),X]_{1-\delta}$.*

From Theorem 2.1, it follows that $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\} \subset \rho(A_q)$. Hence the fractional powers of $I - A_q$ are well defined, closed, linear, invertible operators and, for any $\delta \in (0,1)$, $D\left((I-A_q)^\delta\right)$ endowed with the norm $\|z\|_\delta \doteq \left\|(I-A_q)^\delta z\right\|_q$ is a Banach space, which we denote with $Z_{q,\delta}$. This space coincides with the interpolation space $[D(A_q), Z_q]_{1-\delta}$. The following result concerning the regularity of F was proved in [32].

Theorem 2.4. ([32]) *Assume* **(H1)** *holds. Let* $q \in \mathcal{Q}$, $0 < \epsilon < \frac{1}{4}$ *and* U *a bounded subset of* $[0,T] \times D\left((I - A_q)^{\frac{3}{4}+\epsilon}\right)$. *Then there exists a constant* $L > 0$ *depending on* U, ϵ *and* q, *such that*

$$\|F(q, t_1, z_1) - F(q, t_2, z_2)\|_q \leq L\left(|t_1 - t_2| + \|z_1 - z_2\|_{\frac{3}{4}+\epsilon}\right)$$

for every (t_1, z_1), $(t_2, z_2) \in U$. *Moreover, the constant* L *can be chosen independent of* q *on compact subsets of* Q.

Observation. The operator $I - A_q$ above can be replaced by $\eta I - A_q$ for any $\eta > 0$ without changing any of the assertions. The choice $\eta = 1$ has no particular meaning.

3. Spectral Approximations

In this section finite-dimensional approximating solutions to problem $(\mathcal{P})$ are defined and their convergence to the exact solution is shown.

In the sequel the parameter $q \in \mathcal{Q}$ will be fixed, so, wherever it is clear from the context, we shall suppress it from the notation.

For fixed $N \in \mathbb{N}$ let

$$\beta_n^N(x) \doteq \begin{pmatrix} \sin \pi n x \\ \lambda_n^+ \sin \pi n x \\ 0 \end{pmatrix}, \quad \beta_{N+n}^N(x) \doteq \begin{pmatrix} \sin \pi n x \\ \lambda_n^- \sin \pi n x \\ 0 \end{pmatrix}, \quad \beta_{2N+n}^N(x) \doteq \begin{pmatrix} 0 \\ 0 \\ \cos \pi (n-1) x \end{pmatrix},$$

for $n = 1, 2, \cdots, N$, where $\lambda_n^{+,-}$ are as in Theorem 2.1, and let us define Z^N to be the span of $\hat{\beta}_N \doteq \left\{\beta_n^N(x)\right\}_{n=1}^{3N}$ endowed with the Z-norm. Then $\bigcup_{N=1}^{\infty} Z^N$ is dense in Z and, since the β_n^N's are eigenvectors of A, it follows that Z^N is invariant under A. Note also that Z^N is itself a Hilbert space.

Next, we define the finite-dimensional approximating problem $\left(\mathcal{P}^N\right)$ in Z^N, as follows.

$$\left(\mathcal{P}^N\right) \begin{cases} \frac{d}{dt} z^N(t) = A^N z^N(t) + F^N(t, z^N(t)), \quad 0 \leq t \leq T \\ z^N(0) = P^N z_0 \end{cases}$$

where $P^N : Z \to Z^N$ is the orthogonal projection of Z onto Z^N, A^N is the restriction of A to Z^N and $F^N(t, z) \doteq P^N F(t, z)$. The density of $\bigcup_{N=1}^{\infty} Z^N$ in Z implies the strong convergence of P^N to the identity. Moreover, a straightforward calculation using the spectral decomposition of A shows that $\bigcup_{N=1}^{\infty} Z^N$ is also dense in $Z_{q,\delta}$ and $\left\|P^N z - z\right\|_\delta \to 0$, $\forall z \in Z_{q,\delta}$.

Since Z^N has finite dimension, the operator A^N on Z^N is bounded and linear, and a fortiori, it generates a uniformly continuous semigroup of bounded linear operators $T^N(t)$ on Z^N.

We have the following result on local existence of solutions of problem $\left(\mathcal{P}^N\right)$.

Theorem 3.1. *Let* $z_0 \in Z$. *Then, for any positive integer N, there exists* $t_1^N > 0$ *such that* $\left(\mathcal{P}^N\right)$ *has a unique solution on* $[0, t_1^N)$.

Proof. Let $\delta \in (\frac{3}{4}, 1)$, $z_0 \in Z$ and $N \in \mathbb{N}$ be fixed. By virtue of Theorem 2.4, there exists a constant $L(r, t')$ such that for any $r > 0$ and $t' > 0$

$$\|F(t, z) - F(s, w)\|_Z \leq L(r, t')\left(|t - s| + \|(I - A)^\delta (z - w)\|_Z\right)$$

for every $t, s \in [0, t']$ and $z, w \in D\left((I - A)^\delta\right)$ with $\|z\|_\delta \leq r$, $\|w\|_\delta \leq r$. Then, for every $t, s \in [0, t']$, and $z, w \in Z^N$ with $\|z\|_\delta \leq r$, $\|w\|_\delta \leq r$ we have

$$\begin{aligned} \|F^N(t, z) - F^N(s, w)\|_{Z^N} &= \left\|P^N\left(F(t, z) - F(s, w)\right)\right\|_Z \\ &\leq \|F(t, z) - F(s, w)\|_Z \\ &\leq L(r, t')\left(|t - s| + \|(I - A)^\delta (z - w)\|_Z\right) \\ &= L(r, t')\left(|t - s| + \|z - w\|_\delta\right) \\ &\leq L(r, t') C(\delta, N)\left(|t - s| + \|z - w\|_{Z^N}\right) \end{aligned}$$

where the constant $C(\delta, N)$ appears because of the equivalence of the norms in Z^N.

Hence, the mapping $(t,z) \to A^N z + F^N(t,z)$ is locally Lipschitz continuous from $[0,T] \times Z^N$ into Z^N and therefore there must exist $t_1^N > 0$ such that problem $\left(\mathcal{P}^N\right)$ has a unique solution on $[0, t_1^N)$. ■

The following result relates the semigroups $T(t)$ and $T^N(t)$.

Lemma 3.2. *Let $T(t)$, $T^N(t)$, A, A^N, Z and Z^N be as above and let $R(\lambda; A)$ denote the resolvent of A at λ, $R(\lambda; A) \doteq (\lambda I - A)^{-1}$. Then*

i) *for every $\lambda \in \rho(A)$, the space Z^N is invariant under $R(\lambda; A)$;*

ii) *the restriction of $T(t)$ to Z^N coincides with $T^N(t)$ for every $t \geq 0$, i.e.*

$$T(t)|_{Z^N} = T^N(t) \quad \forall t \geq 0.$$

Proof. **i)** Let $\lambda \in \rho(A)$ and ξ be an element of the basis $\hat{\beta}_N$ of Z^N and define $z \doteq R(\lambda; A)\xi$. Then z is an eigenvector of A corresponding to the same eigenvalue σ of ξ. In fact, $(\lambda I - A)Az = A\xi = \sigma\xi$, which implies $Az = R(\lambda; A)\sigma\xi = \sigma z$. Since all the eigenvalues of A are simple, z must be a constant multiple of ξ and therefore $z \in Z^N$. Part **i)** then follows by the linearity of $R(\lambda; A)$.
ii) Since Z^N is invariant under A, the operator $\tilde{A}^N$, the part of A in Z^N, defined by

$$\begin{aligned} D(\tilde{A}^N) &\doteq \left\{ z \in D(A) \cap Z^N : Az \in Z^N \right\} \\ \tilde{A}^N z &\doteq Az, \quad z \in D\left(\tilde{A}^N\right), \end{aligned}$$

coincides with A^N, the restriction of A to Z^N. Hence $\tilde{A}^N$ generates a uniformly continuous semigroup on Z^N and by part **i)**, Z^N is invariant under $R(\lambda; A)$ for every λ with Re $\lambda > 0$. By Theorem 4.5.5 in [26] it follows that $\tilde{A}^N = A^N$ is the infinitesimal generator of the restriction of $T(t)$, the semigroup generated by A, to Z^N. ■

We shall need the following generalization of Gronwall's Lemma for singular kernels whose proof can be found in [17, Lemma 7.1.1].

Lemma 3.3. ([17]) *Let $a(t)$ be a nonnegative, locally integrable function on $0 \leq t \leq T$, $L \geq 0$ and $0 < \delta < 1$. Then, there exists a constant $K = K(\delta)$ such that every function u satisfying*

$$u(t) \leq a(t) + L \int_0^t \frac{1}{(t-s)^\delta} u(s)\, ds$$

on $0 \leq t \leq T$, also satisfies

$$u(t) \leq a(t) + KL \int_0^t \frac{a(s)}{(t-s)^\delta}\, ds, \qquad \textit{for } 0 \leq t < T.$$

We define the operator $A_I \doteq A - I$ with $D(A_I) = D(A)$. From the properties of A it follows easily that A_I generates an exponentially stable analytic semigroup $T_I(t)$. Moreover, $T_I(t) = e^{-t}T(t)$. Also, since $0 \in \rho(A_I)$, the fractional powers $(-A_I)^\delta$ are well defined for any $\delta \in (0,1)$.

We now proceed to state and prove our main result about the convergence of the approximating solutions.

Theorem 3.4. *Let $\delta \in \left(\frac{3}{4}, 1\right)$, $z_0 \in D\left((-A_I)^\delta\right)$ and suppose $z^N(t)$, $z(t)$ are solutions of $\left(\mathcal{P}^N\right)$ and $(\mathcal{P})$, respectively, and let $[0, t_1)$ be the maximal interval of existence of $z(t)$. Then, for any $t_1' < t_1$ there exists a constant N_0 such that $z^N(t)$ exists on $[0, t_1']$ for every $N \geq N_0$ and $z^N(t)$ converges to $z(t)$ in the norm of Z for every $t \in [0, t_1']$. Moreover, the convergence holds in the norm of the graph of $(-A_I)^\delta$.*

Proof. Let $\delta \in \left(\frac{3}{4}, 1\right)$, $t_1' < t_1$ and for each $N \in \mathbb{N}$ let $t_1^N > 0$ be such that $z^N(t)$ exists on $[0, t_1^N)$. Then, for $t \in [0, \min\{t_1', t_1^N\})$ and $N \in \mathbb{N}$

$$\begin{aligned} z^N(t) &= T^N(t)P^N z_0 + \int_0^t T^N(t-s)P^N F(s, z^N(s))\, ds, \\ z(t) &= T(t)z_0 + \int_0^t T(t-s)F(s, z(s))\, ds. \end{aligned}$$

Therefore

$$\begin{aligned}\|z^N(t)-z(t)\|_\delta &= \left\|(-A_I)^\delta\left(z^N(t)-z(t)\right)\right\|\\ &\le \left\|(-A_I)^\delta\left(T^N(t)P^Nz_0-T(t)z_0\right)\right\|\\ &\quad+\int_0^t \left\|(-A_I)^\delta\left[T^N(t-s)P^NF(s,z^N(s))-T(t-s)F(s,z(s))\right]\right\|\,ds\\ &\doteq \rho_1^N(t)+\rho_2^N(t).\end{aligned}$$

Since $T(t)$ commutes with $(-A_I)^\delta$,

$$\begin{aligned}\rho_1^N(t) &= \left\|(-A_I)^\delta T(t)\left(P^Nz_0-z_0\right)\right\|_Z\\ &\le \|T(t)\|_{\mathcal{L}(Z)}\left\|(-A_I)^\delta\left(P^Nz_0-z_0\right)\right\|_Z\\ &\le \left\|P^Nz_0-z_0\right\|_\delta.\end{aligned}$$

Similarly, for the integrand defining $\rho_2^N(t)$ we have

$$\begin{aligned}\big\|(-A_I)^\delta\big[T^N(t-s)\ &P^NF(s,z^N(s))-T(t-s)F(s,z(s))\big]\big\|_Z\\ &= \left\|(-A_I)^\delta T(t-s)\left(P^NF(s,z^N(s))-F(s,z(s))\right)\right\|_Z\\ &\le \left\|(-A_I)^\delta T(t-s)\right\|_{\mathcal{L}(Z)}\left\|P^NF(s,z^N(s))-F(s,z(s))\right\|_Z\\ &\le e^{t_1'}\frac{C_\delta}{(t-s)^\delta}\left\|P^NF(s,z^N(s))-F(s,z(s))\right\|_Z. \end{aligned}\tag{3.1}$$

But

$$\begin{aligned}\left\|P^NF(s,z^N(s))-F(s,z(s))\right\|_Z &\le \left\|P^N\left[F(s,z^N(s))-F(s,z(s))\right]\right\|_Z+\left\|\left(P^N-I\right)F(s,z(s))\right\|_Z\\ &\le \left\|F(s,z^N(s))-F(s,z(s))\right\|_Z+\left\|\left(P^N-I\right)F(s,z(s))\right\|_Z.\end{aligned}\tag{3.2}$$

Hence, from (3.1) and (3.2) it follows that

$$\begin{aligned}\rho_2^N(t) &\le C_\delta e^{t_1'}\int_0^t\frac{1}{(t-s)^\delta}\left\|F(s,z^N(s))-F(s,z(s))\right\|_Z\,ds\\ &\quad+C_\delta e^{t_1'}\int_0^t\frac{1}{(t-s)^\delta}\left\|(P^N-I)F(s,z(s))\right\|_Z\,ds.\end{aligned}$$

The integrand of the second term on the RHS above is bounded by $\dfrac{2}{(t-s)^\delta}\|F(s,z(s))\|_Z\in L^1(0,T)$, uniformly in N, and converges to zero as N tends to infinity. By the Dominated Convergence Theorem the second term of the last inequality tends to zero as N goes to infinity.

Summarizing, we have

$$\|z^N(t)-z(t)\|_\delta \le \epsilon^N(t)+\int_0^t\frac{\tilde{C}}{(t-s)^\delta}\left\|F(s,z^N(s))-F(s,z(s))\right\|\,ds \tag{3.3}$$

where, for $t\in[0,t_1']$, $\epsilon^N(t)\le C$ for all $N\in\mathbb{N}$ and $\epsilon^N(t)\to 0$ as $N\to\infty$. In particular $\displaystyle\int_0^{t_1'}\epsilon^N(t)\,dt\to 0$ as $N\to\infty$.

Let $K=K(\delta)$ be as in Lemma 3.3 and let us define $\tilde{K}\doteq C+C\tilde{C}K$. Since $z^N(0)=P^Nz_0$ there exists $\delta^N>0$ such that $\left\|z^N(t)\right\|_\delta\le M+2\tilde{K}$ for all $t\in[0,\delta^N]$, where $M\doteq\sup_{0\le t\le t_1'}\|z(t)\|_\delta$. Let L be the Lipschitz constant for F corresponding to the set $U\doteq[0,t_1']\times\left\{\|z\|_\delta\le M+2\tilde{K}\right\}$. Then, from (3.3)

$$\left\|z^N(t)-z(t)\right\|_\delta\le\epsilon^N(t)+\tilde{C}L\int_0^t\frac{1}{(t-s)^\delta}\left\|z^N(s)-z(s)\right\|_\delta\,ds\qquad\forall t\in[0,\delta^N],$$

and, from Lemma 3.3

$$\left\|z^N(t) - z(t)\right\|_\delta \le f^N(t), \quad \forall t \in [0, \delta^N], \tag{3.4}$$

where $f^N(t) \doteq \epsilon^N(t) + \tilde{C}KL \int_0^t \frac{\epsilon^N(s)}{(t-s)^\delta}\, ds$, for $t \in [0, t_1']$.

We shall now show that there exists $N_0 \in \mathbb{N}$ such that $f^N(t) \le \tilde{K}$ $\forall t \in [0, t_1']$, $\forall N \ge N_0$. As we shall later see this will imply not only the existence of $z^N(t)$ on the whole interval $[0, t_1']$, $\forall N \ge N_0$, but also the bound $\|z^N(t)\|_\delta \le M + 2\tilde{K}$, $\forall t \in [0, t_1']$, $\forall N \ge N_0$.

In fact, observe that

$$\begin{aligned}\int_0^t \frac{\epsilon^N(t)}{(t-s)^\delta}\, ds &\le \int_0^t \frac{C}{(t-s)^\delta}\, ds \\ &\le \int_0^t C \frac{1}{s^\delta}\, ds \\ &= \frac{C}{1-\delta} t^{1-\delta}.\end{aligned}$$

Choosing $\eta = \eta(L) > 0$ sufficiently small so that $t^{1-\delta} \le \dfrac{1-\delta}{2L}$ for every $t \in [0, \eta]$, it follows that

$$\int_0^t \frac{\epsilon^N(t)}{(t-s)^\delta} \le \frac{C}{2L} \quad \text{for every } t \in [0, \eta]. \tag{3.5}$$

On the other hand, if $\eta < t \le t_1'$

$$\begin{aligned}\int_0^t \frac{\epsilon^N(t)}{(t-s)^\delta}\, ds &= \int_0^t \frac{\epsilon^N(t-s)}{s^\delta}\, ds \\ &= \int_0^\eta \frac{\epsilon^N(t-s)}{s^\delta}\, ds + \int_\eta^t \frac{\epsilon^N(t-s)}{s^\delta}\, ds \\ &\le \frac{C}{1-\delta}\eta^{1-\delta} + \int_\eta^t \frac{\epsilon^N(t-s)}{\eta^\delta}\, ds \\ &\le \frac{C}{2L} + \frac{1}{\eta^\delta} \int_\eta^t \epsilon^N(t-s)\, ds \\ &\le \frac{C}{2L} + \frac{1}{\eta^\delta} \int_0^{t_1'} \epsilon^N(s)\, ds.\end{aligned}$$

Then, since $\displaystyle\int_0^{t_1'} \epsilon^N(s)\, ds \to 0$, then there exists N_0 such that

$$\int_0^t \frac{\epsilon^N(t)}{(t-s)^\delta}\, ds \le \frac{C}{L} \quad \forall t \in [\eta, t_1'] \text{ and } N \ge N_0. \tag{3.6}$$

From (3.5) and (3.6) it follows that

$$f^N(t) \le C + C\tilde{C}K = \tilde{K} \quad \forall t \in [0, t_1'] \text{ and } N \ge N_0, \tag{3.7}$$

as wanted.

Consequently, from (3.4) and (3.7)

$$\left\|z^N(t) - z(t)\right\|_\delta \le \tilde{K} \quad \forall N \ge N_0 \text{ and } t \in [0, \delta^N],$$

which implies

$$\left\|z^N(t)\right\|_\delta \le M + \tilde{K} \quad \forall N \ge N_0 \text{ and } t \in [0, \delta^N].$$

Now, let $N \ge N_0$ be fixed. Then $z^N(t)$ exists on $[0, t_1']$ and for $t \in [0, t_1']$, $\left\|z^N(t)\right\|_\delta \le M + 2\tilde{K}$. In fact, suppose, on the contrary, that there exists $t^* < t_1'$ such that $\left\|z^N(t^*)\right\|_\delta = M + 2\tilde{K}$ and $\left\|z^N(t)\right\|_\delta <$

$M+2\tilde{K}$ for $0 \le t < t^*$. Then, from (3.4) $\left\|z^N(t)-z(t)\right\|_\delta \le f^N(t) \le \tilde{K}$ on $[0,t^*)$, and therefore, $\left\|z^N(t)\right\|_\delta \le M+\tilde{K}$ on $[0,t^*)$. By the continuity of $z^N(t)$, we must have $\left\|z^N(t^*)\right\|_\delta \le M+\tilde{K}$, which contradicts $\left\|z^N(t^*)\right\|_\delta = M+2\tilde{K}$.

At this point we have shown that $\forall N \ge N_0$, $\|z^N(t)\|_\delta \le M+2\tilde{K}$, $\forall t \in [0,t_1']$. Therefore, for $N \ge N_0$, δ^N can be chosen strictly greater than t_1'. Hence (3.4) holds in $[0,t_1']$, i.e.

$$\left\|z^N(t)-z(t)\right\|_\delta \le f^N(t), \quad \forall t \in [0,t_1'].$$

Finally, since by virtue of the Dominated Convergence Theorem, $f^N(t) \to 0$ $\forall t \in [0,t_1']$ as $N \to \infty$, the theorem follows. ■

4. Time Discretization

In this section we shall first find the representation of the approximating problem $\left(\mathcal{P}^N\right)$ in the basis $\hat{\beta}_N$ of Z^N. For this purpose, let w^N be the vector whose components are the coefficients of the solution $z^N(t) = \begin{pmatrix} u^N(t) \\ v^N(t) \\ \theta^N(t) \end{pmatrix}$ of problem $\left(\mathcal{P}^N\right)$ in the basis $\hat{\beta}_N$. Then $w^N(t)$ is the solution of the IVP

$$\left(\tilde{\mathcal{P}}^N\right) \begin{cases} \dot{w}^N(t) = \tilde{A}^N w^N(t) + \tilde{F}^N\left(t, w^N(t)\right) \\ w^N(0) = \gamma^N \end{cases}$$

with

$$\tilde{A}^N = \left(Q^N\right)^{-1} K^N, \qquad \tilde{F}^N(t,w) = \left(Q^N\right)^{-1} R^N F\left(t, Q^N w\right), \qquad \gamma^N = \left(Q^N\right)^{-1} R^N \begin{pmatrix} u_0 \\ u_1 \\ \theta_0 \end{pmatrix},$$

where the matrices Q^N, K^N and the mapping $R^N : Z^N \to \mathbb{R}^{3N}$ are defined by

$$\left(Q^N\right)_{i,j} = \left\langle \beta_i^N, \beta_j^N \right\rangle_q, \qquad \left(K^N\right)_{i,j} = \left\langle \beta_i^N, A^N \beta_j^N \right\rangle_q, \qquad \left(R^N z\right)_i = \left\langle \beta_i^N, z \right\rangle_q,$$

$i,j = 1,2,\cdots,3N$.

For the time discretization, the following hybrid implicit-explicit Euler method was used:

$$\begin{aligned} w_0^N &= \gamma^N \\ \frac{1}{\Delta t}\left(w_{k+1}^N - w_k^N\right) &= \tilde{A}^N w_{k+1}^N + \tilde{F}^N\left(k\Delta t, w_k^N\right), \qquad k = 0,1,\cdots. \end{aligned}$$

The convergence of the solutions of the corresponding time-discretized system to the solution of $\left(\tilde{\mathcal{P}}^N\right)$ as $\Delta t \to 0$ follows immediately from the following theorem.

Theorem 4.1. *Let n be a positive integer, $B \in \mathcal{L}(\mathbb{R}^n)$, $G : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function such that $G(t,\cdot)$ is Lipschitz continuous in $\mathbb{R}^n$ for any $t \in [0,T]$, with constant K independent of t. Assume that the IVP*

$$\begin{cases} W'(t) = BW(t) + G(t, W(t)), \quad W(t) \in \mathbb{R}^n, \\ W(0) = W_0 \end{cases}$$

has a unique solution $W(\cdot)$ in $[0,T]$ such that $W''(t)$ is bounded in $[0,T]$. Given $0 < h < 1/\|B\|$ and $w_0 \in \mathbb{R}^n$ define $N(h) = [T/h]$ (the integer part of N/h) and given w_0, define w_j, $j = 1,\cdots,N(h)$, by

$$w_{j+1} = w_j + hBw_{j+1} + hG(jh, w_j), \qquad j = 0,1,\cdots,N(h)-1.$$

Then, there exists a constant $\tilde{C}$ such that

$$\max_{j=0,1,\cdots,N(h)} |W(jh) - w_j| \le e^{T\left(\frac{K+\|B\|}{1-h\|B\|}\right)} |W_0 - w_0| + h\tilde{C}\left[e^{T\left(\frac{K+\|B\|}{1-h\|B\|}\right)} - 1\right].$$

If, moreover, $h < 1/2\|B\|$ then

$$\max_{j=0,1,\cdots,N(h)} |W(jh) - w_j| \leq e^{2T(K+\|B\|)} |W_0 - w_0| + h\tilde{C}\left[e^{2T(K+\|B\|)} - 1\right].$$

Remark: It is not difficult to show that the conclusions of this theorem remain valid under the weaker assumptions G locally Lipschitz and W' Lipschitz continuous.

Proof. Let $0 < h < 1/\|B\|$ and define $W_j = W(jh)$. Using the Taylor approximation theorem

$$\begin{aligned} W_{j+1} &= W_j + hW'(jh) + \frac{h^2}{2}W''(\xi_{j,h}) \\ &= W_j + hBW_j + hG(jh, W_j) + \frac{h^2}{2}W''(\xi_{j,h}). \end{aligned}$$

where $\xi_{j,h} \in (jh, (j+1)h)$. Defining $e_j = W_j - w_j$ we then have

$$\begin{aligned} e_{j+1} &= e_j + hB(W_j - w_{j+1}) + h[G(jh, W_j) - G(jh, w_j)] + \frac{h^2}{2}W''(\xi_{j,h}) \\ &= e_j + hBe_{j+1} + h[G(jh, W_j) - G(jh, w_j)] + hB(W_j - W_{j+1}) + \frac{h^2}{2}W''(\xi_{j,h}), \end{aligned}$$

or equivalently

$$e_{j+1} - hBe_{j+1} = e_j + h[G(jh, W_j) - G(jh, w_j)] + hB(W_j - W_{j+1}) + \frac{h^2}{2}W''(\xi_{j,h}).$$

Therefore,

$$\begin{aligned} (1 - h\|B\|)|e_{j+1}| &\leq |e_j| + h|G(jh, W_j) - G(jh, w_j)| + h\|B\||W_j - W_{j+1}| + \frac{h^2}{2}|W''(\xi_{j,h})| \\ &\leq |e_j| + hK|e_j| + h^2\|B\|\left|W'\left(\tilde{\xi}_{j,h}\right)\right| + \frac{h^2}{2}|W''(\xi_{j,h})| \end{aligned}$$

where $\tilde{\xi}_{j,h} \in (jh, (j+1)h)$.

Letting C be an upper bound for $|W'(t)| + |W''(t)|$ in $[0, T]$, it follows that

$$|e_{j+1}| \leq \frac{1 + hK}{1 - h\|B\|}|e_j| + \frac{h^2 C(1 + \|B\|)}{1 - h\|B\|}.$$

By induction one has

$$|e_j| \leq \left[\frac{1 + hK}{1 - h\|B\|}\right]^j |e_0| + \frac{h^2 C(1 + \|B\|)}{1 - h\|B\|} \frac{\left[\frac{1+hK}{1-h\|B\|}\right]^j - 1}{\frac{1+hK}{1-h\|B\|} - 1}.$$

Now, since $(1 + x)^j \leq e^{jx}$ for any $x > -1$ and $j \in \mathbb{N}$, we have that

$$\left[\frac{1 + hK}{1 - h\|B\|}\right]^j = \left[1 + h\frac{K + \|B\|}{1 - h\|B\|}\right]^j \leq e^{jh\left(\frac{K+\|B\|}{1-h\|B\|}\right)} \leq e^{T\left(\frac{K+\|B\|}{1-h\|B\|}\right)}$$

and

$$\frac{\left[\frac{1+hK}{1-h\|B\|}\right]^j - 1}{\frac{1+hK}{1-h\|B\|} - 1} = \frac{\left[\frac{1+hK}{1-h\|B\|}\right]^j - 1}{h\frac{K+\|B\|}{1-h\|B\|}}$$

Hence

$$|W_j - w_j| \leq e^{T\left(\frac{K+\|B\|}{1-h\|B\|}\right)}|W_0 - w_0| + hC\frac{1 + \|B\|}{K + \|B\|}\left[e^{T\left(\frac{K+\|B\|}{1-h\|B\|}\right)} - 1\right]$$

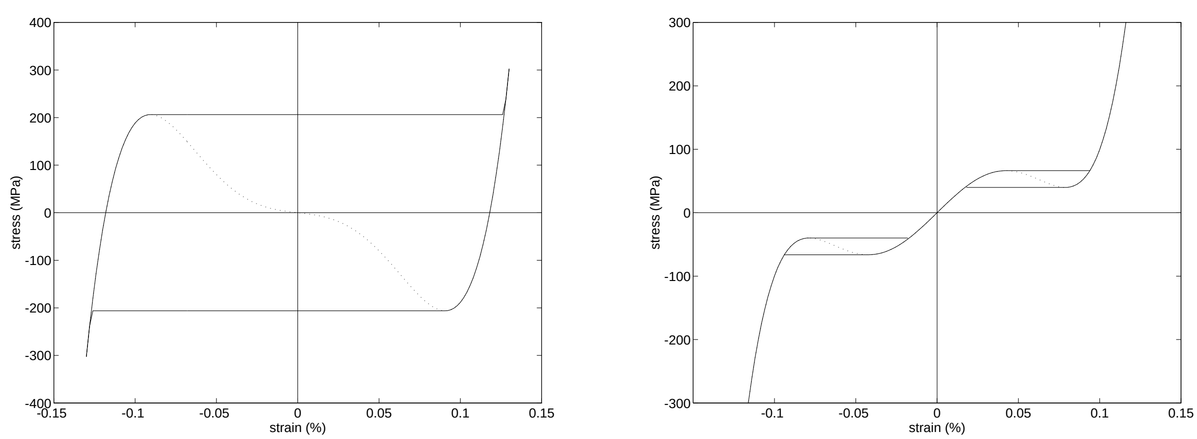
for any $j = 0, 1, \dots, N(h)$. This proves the first part of the theorem. The final assertion follows from the fact that $1 - h\|B\| > 1/2$ if $h < 1/2\|B\|$. ■

Comment: The previous theorem together with the result of Theorem 3.4 ensures the convergence of the fully-discretized system to the solutions of (2.1) as $N \to \infty$ and $\Delta t \to 0$ in an appropriate way. Moreover, for fixed N, Theorem 4.1 shows that the order of convergence of the time-discretized equations to the solution of $\left(\tilde{\mathcal{P}}^N\right)$ is $O(\Delta t)$. However, we have not obtained an order of convergence for the fully-discretized system. Moreover, due to the intrinsic nature of the spectral approximations being used, it is not expected that such an order of convergence could be obtained.

5. Numerical Results

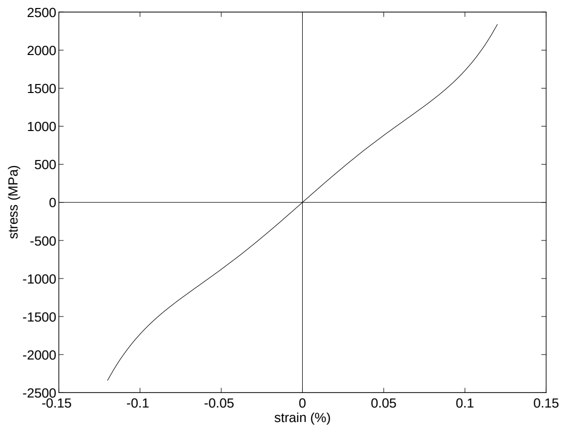
For the numerical results presented below we used this hybrid method with $N = 32$ and $\Delta t = 10^{-5}$ and the parameter values reported by F. Falk in [13] for the alloy $Au_{23}Cu_{30}Zn_{47}$: $\alpha_2 = 24\, J\, cm^{-3}\, K^{-1}$, $\alpha_4 = 1.5{\times}10^5\, J\, cm^{-3}$, $\alpha_6 = 7.5{\times}10^6\, J\, cm^{-3}\, K^{-1}$, $\theta_1 = 208\, K$, $C_v = 2.9\, J\, cm^{-3}\, K^{-1}$, $k = 1.9\, w\, cm^{-1}\, K^{-1}$, $\rho = 11.1\, g\, cm^{-3}$. We also took $\gamma = 10^{-12}\, J\, cm^{-1}$ as reported in [15]. For the value of β we chose $\beta = 1$. This choice has no particular physical meaning. To our knowledge, there are no reports of values of β for real materials although there seems to be evidence that for some SMA, β is either very small or zero. Figure 1 shows the stress-strain curves obtained from the potential (1.2) for these values of the parameters. The doted lines indicate the unstable parts of the curves, while the horizontal lines indicate possible hysteresis loops.

It is important to mention that the results in this article depend stronly on the hypothesis $\beta > 0$, that is on the assumption that there are viscous stresses in the material. This approach does not work if $\beta = 0$. For the non-viscous case $\beta = 0$ we refer the reader to [30], [34], [29]. An algorithm for the numerical approximation of the solutions in this case was introduced by Niezgodka and Sprekels in [25]. Further numerical and stability results can be found in [18].



(a) $\theta = 200^0$ K

(b) $\theta = 260^0$ K



(c) $\theta = 600^0$ K

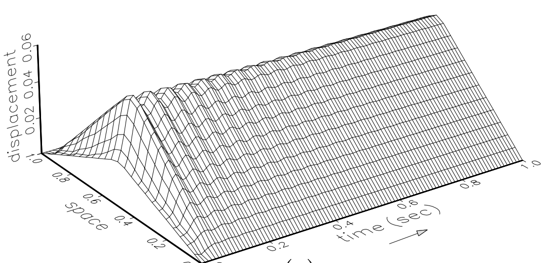
Figure 1: Stress-Strain curves for different temperatures obtained from (1.2) with the values of α_2, α_4, α_6 and θ_1 as in [13]. The dotted lines represent unstable parts of the curves. Horizontal lines indicate possible hysteresis loops.

Experiment 1: *Low-temperature steady-state.*

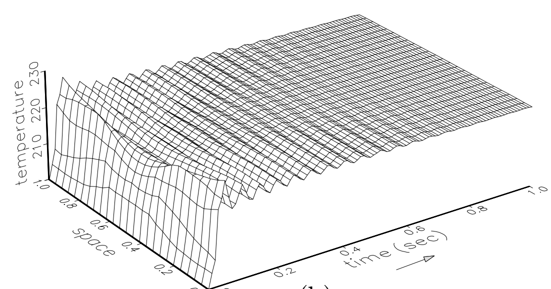
For this experiment we took $f = g \equiv 0$, $\theta_0(x) \equiv 200^0$ K, $u_0(x) = P^N h(x)$, where

$$h(x) = \begin{cases} 0.05x, & \text{if } 0 \leq x \leq 0.5, \\ 0.05(1-x), & \text{if } 0.5 \leq x \leq 1, \end{cases}$$

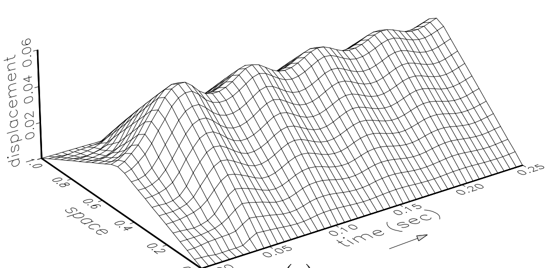
and $v_0 \equiv 0$. Thus, the beam is initially in the low temperature range composed of two segments of martensites, namely, martensite M_+ on $0 \leq x < \frac{1}{2}$ and martensite M_- on $\frac{1}{2} < x \leq 1$ (5% initial strain). The evolution of displacement and temperature can be observed in Figures 2a and 2b, respectively. This evolution is due to the fact that the initial condition $z_0(x) = \begin{pmatrix} u_0(x) \\ v_0(x) \\ \theta_0(x) \end{pmatrix}$ does not correspond to a steady-state of system (1.1a-e). The system evolves until a steady-state composed of two symmetric segments of martensites M_+ and M_- ($\cong 11.25\%$ strain) and constant temperature $\theta \cong 222^0$ K is reached. Figure 2c shows in more detail the displacement profile during the first 250 milliseconds.



(a)



(b)



(c)

Figure 2: *Low temperature steady-state.* Evolution of displacement (a, c) and temperature (b) from an unsteady low temperature initial condition.

Experiment 2: *High-temperature steady-state.*

Here we took $\theta_0(x) \equiv 600^0$ K and u_0, v_0, f and g as in Experiment 1. The evolution of displacement and temperature is shown in Figures 3a and 3b, respectively. The beam oscillates until the steady-state consisting of zero deformation and constant temperature $\theta \cong 505.6^0$ K is reached. This is in agreement with the fact that above the austenite-finish temperature $\theta = A_f$ (in this case $A_f \cong 283^0$ K) the steady-states satisfy $u \equiv 0$, $\theta \equiv const.$ Due to the high-temperature unsteady initial condition the beam immediately bends downward approaching the state $u \equiv 0$ while temperature decreases slightly, originating the damped oscillations observed in Figures 3a and 3b. The oscillations of the middle-point of the beam can be appreciated in Figure 3c.

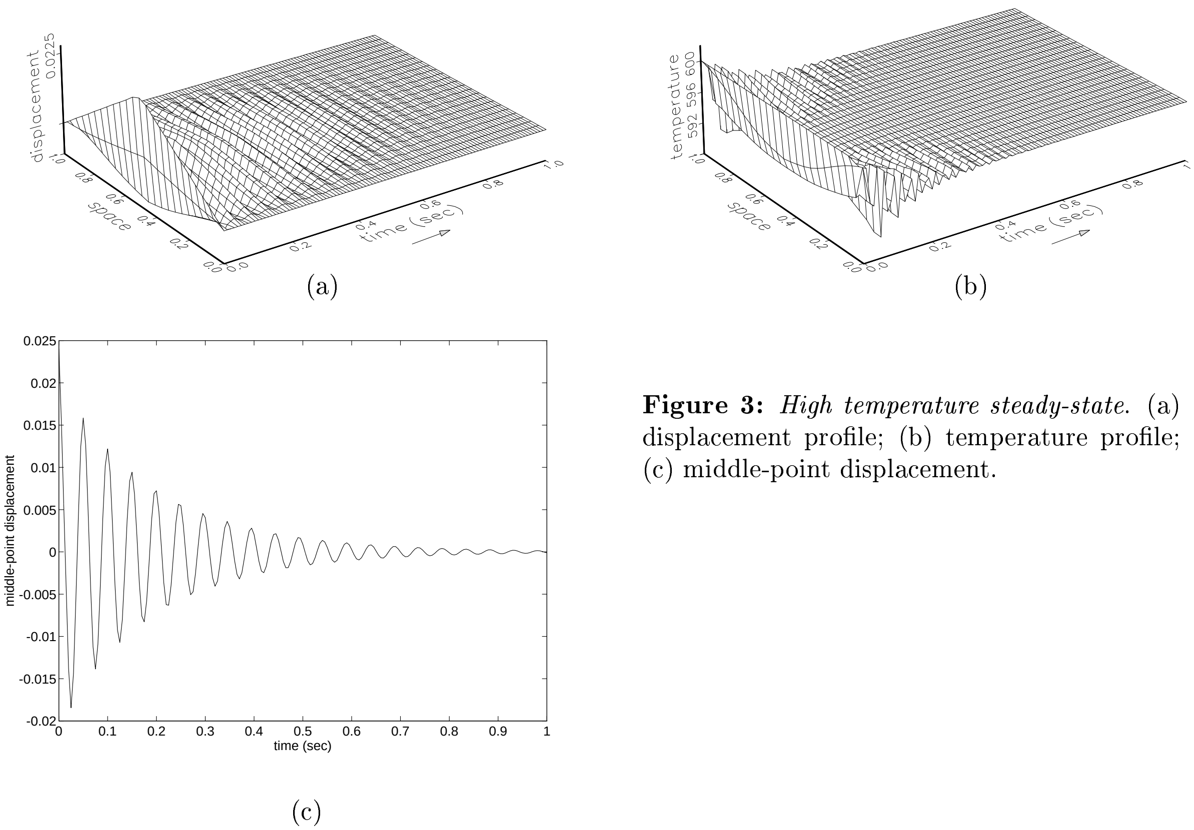


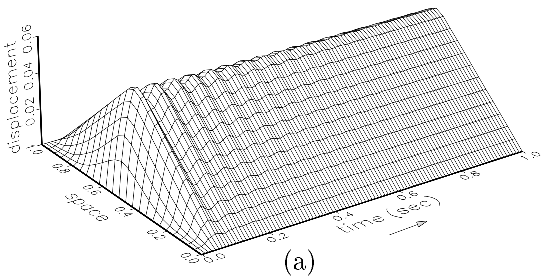
Figure 3: *High temperature steady-state.* (a) displacement profile; (b) temperature profile; (c) middle-point displacement.

Experiment 3: *Pulse at low temperature.*

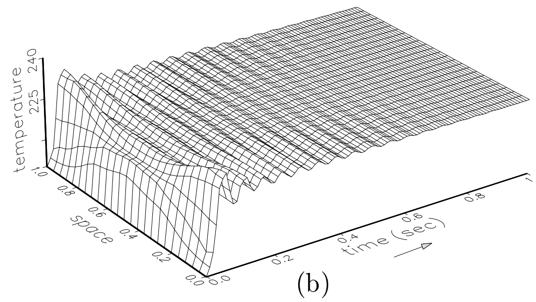
In this experiment we studied the effects of a distributed force consisting of a pulse around the middle-point of the beam when the initial temperature is below the martensite finish temperature $\theta = M_f \cong 208^0$ K. We took $u_0(x) = v_0(x) \equiv 0$, $\theta_0(x) \equiv 200^0$ K, $g(x,t) \equiv 0$ and

$$f(x,t) = \begin{cases} 5 \times 10^4, & \text{if } 0.4 \leq x \leq 0.6 \text{ and } 0 < t < 0.5 \times 10^{-3}, \\ 0, & \text{otherwise.} \end{cases}$$

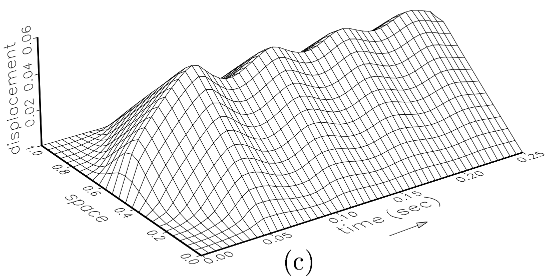
Initially, points around the center move upward while the effect of the pulse propagates to the endpoints of the beam (Figures 4a, 4c). At exactly the time this effect reaches the endpoints, the middle-point deflection reaches a maximum and small damped oscillations begin to take place (Figure 4c) around the final equilibrium state consisting of two symmetric segments of martensites M$_+$, M$_-$ ($\cong$ 11.05% strain) and constant temperature $\theta \cong 226^0$ K (Figure 4b).



(a)



(b)

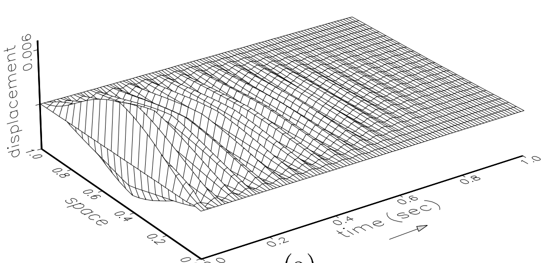


(c)

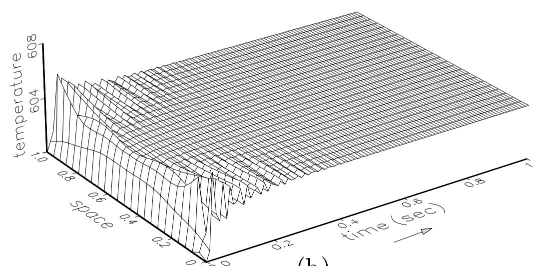
Figure 4: *Pulse at low temperature.* (a), (c) displacement profile; (b) temperature profile.

Experiment 4: *Pulse at high temperature.*

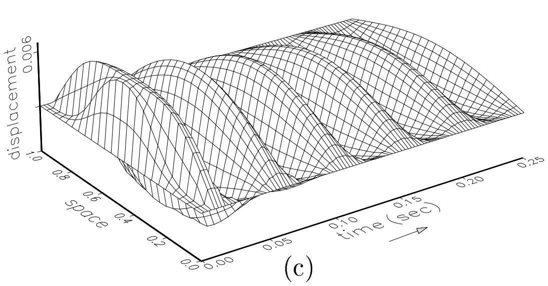
In this case we investigated the effects of a pulse around the middle-point of the beam, which was set initially at a constant temperature above A_f. We took $\theta_0(x) \equiv 600^0$ K and u_0, v_0, f and g as in Experiment 3. At the beginning, the beam bends upward until the pulse is switched off (Figure 5c). Immediately afterwards, damped oscillations begin to occur. These oscillations take place around the final equilibrium state consisting of $u \equiv 0$ and constant temperature $\theta \cong 602^0$ K (Figures 5a, 5b). Recall that above the austenite finish temperature the only unloaded steady-state is $u \equiv 0$.



(a)



(b)



(c)

Figure 5: *Pulse at high temperature.* (a), (c) displacement profile; (b) temperature profile.

Experiment 5: *Waiting-Heating.*

Here, we observed the effects of heating the beam when it is set initially at an equilibrium state consisting of two symmetric segments of martensites M_+ and M_-. For this, we took as the initial data the final steady-state of Experiment 1 (11.25% initial strain, $\theta_0(x) \equiv 222^0$ K), $f(x,t) \equiv 0$ and the heat source $g(x,t)$ consisting of a uniformly spatially distributed heat pulse as follows

$$g(x,t) = \begin{cases} 5 \times 10^4, & \text{if } 0.2 < t < 0.25, \\ 0, & \text{otherwise.} \end{cases}$$

The system remains at the initial state until the heat pulse is switched on. At this time the temperature starts to increase (Figure 6b), the martensite crystals are converted into austenite and the beam bends downward showing small damping oscillations around zero deformation (Figure 6a). These oscillations are quickly damped and the beam reaches the steady-state $u \equiv 0$, $\theta \cong 336^0$ K.

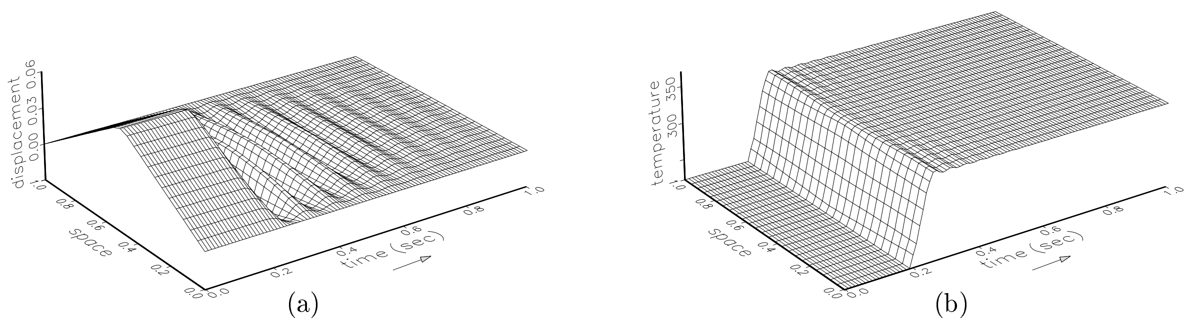


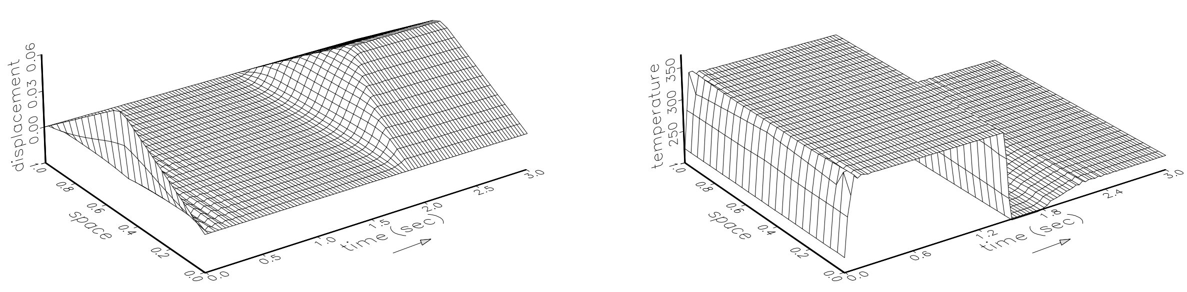
Figure 6: *Waiting-Heating.* (a) displacement and (b) temperature profiles.

Experiment 6: *Heating-Waiting-Cooling (Two-way shape memory effect)*

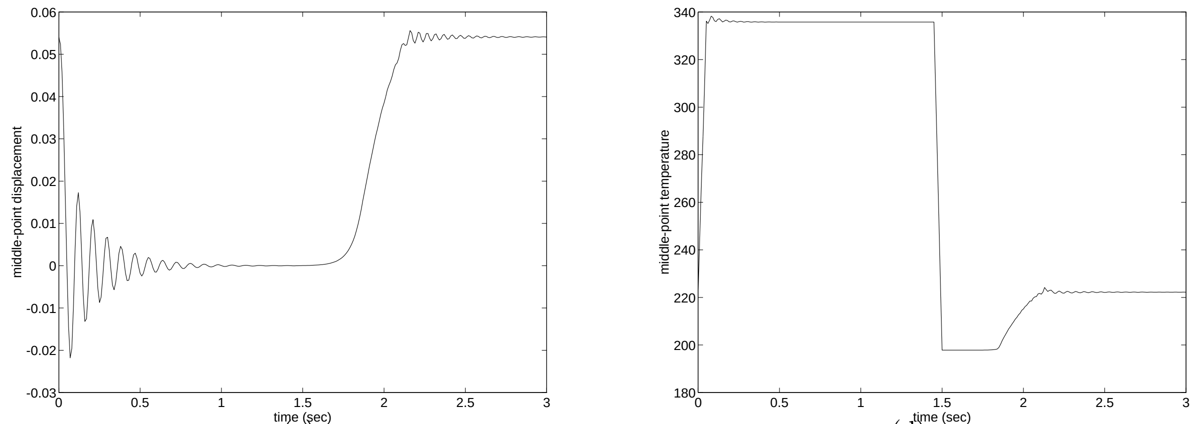
For this experiment we took again as initial data the final steady-state of Experiment 1. We also took $f(x,t) \equiv 0$ and the distributed heat source $g(x,t)$ consisting of an initial uniformly distributed heat pulse which is switched off after $t = 0.05$ sec. At $t = 1.45$ sec. the opposite heat pulse is applied until $t = 1.50$ sec. when it is switched off. More precisely,

$$g(x,t) = \begin{cases} 8 \times 10^3, & \text{if } t < 0.05, \\ -8 \times 10^3, & \text{if } 1.45 < t < 1.50, \\ 0, & \text{otherwise.} \end{cases}$$

The temperature raises uniformly up to nearly 336^0 K while the beam approaches the undeformed state. After the heat pulse is switched off, temperature remains at about 336^0 K while displacement shows small damped oscillations around $u \equiv 0$ due to inertial effects. The sample is now completely in the austenite phase. At $t = 1.45$, when the opposite pulse is applied, the temperature decreases uniformly and remains at about 222^0 K, while the beam undergoes a process of reverse transformation which takes it back to the original initial configuration showing the so-called two-way shape memory phenomenon (Figures 7a-d).



(a) (b)



(c) (d)

Figure 7: *Heating-Waiting-Cooling.* (a) displacement profile; (b) temperature profile; (c) middle-point displacement; (d) middle-point temperature.

6. Conclusions

In this article, discrete spectral approximations to the nonlinear partial differential equations that model the dynamics of thermomechanical martensitic transformations in one-dimensional shape memory alloys with non-convex Landau-Ginzburg potentials were developed.

By using the theories of analytic semigroups and interpolation spaces and a generalization of Gronwall's lemma for singular kernels, the convergence of the approximations was shown to hold not only in the state-space norm but also in the stronger $\|\cdot\|_\delta$-norm.

The numerical experiments performed using this scheme show that under different initial conditions and distributed external actions the model (1.1) is able to produce solutions whose qualitative behavior is found to be in close agreement with laboratory experiments performed on Shape Memory Alloys under similar conditions.

From a practical point of view it would be very important to find the values of the vector parameter q that "best fit" experimental data for a given alloy. This is called the parameter identification problem about which no results are yet known. In this regard the scheme presented here provides a friendly mathematical framework for attacking this problem. Efforts in this direction are already underway and results will be published in a forthcoming article.

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