A posteriori error estimates for elliptic problems with Dirac measure terms in weighted spaces

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Abstract

In this article we develop a posteriori error estimates for general second order elliptic problems with point sources in two- and three-dimensional domains. We prove a global upper bound and a local lower bound for the error measured in a weighted Sobolev space. The weight considered is a (positive) power of the distance to the support of the Dirac delta source term, and belongs to the Muckenhoupt's class A_2. The theory hinges on local approximation properties of either Clément or Scott-Zhang interpolation operators, without need of suitable modifications, and makes use of weighted estimates for fractional integrals and maximal functions. Numerical experiments with an adaptive algorithm yield optimal meshes and very good effectivity indices.

Keywords: elliptic problems, point sources, a posteriori error estimates, finite elements, weighted Sobolev spaces

Mathematics Subject Classification (2000): 35J15, 65N12, 65N15, 65N30, 65N50, 65Y20

1 Introduction

The main goal of this article is to develop a posteriori error estimates for elliptic second order partial differential equations over two- and three-dimensional domains with point sources. Elliptic problems with Dirac measure source terms arise in modeling different applications as, for instance, the electric field generated by a point charge, the acoustic monopoles or pollutant transport and degradation in an aquatic media where, due to the different scales involved, the pollution source is modeled as supported on a single point [1]. Other applications involve the coupling between reaction-diffusion problems taking place in domains of different dimension, which arise in tissue perfusion models [7].

In spite of the fact that the solution of one such problem typically does not belong to H^1, it can be numerically approximated by standard finite elements, but there is no obvious choice for the norm to measure the error. Babuška [3] and Scott [21] obtained a priori estimates for the error measured in L^2 and in fractional Sobolev norms H^s, for s in some subinterval of $(0,1)$, depending on the dimension of the underlying domain. Eriksson [9] showed optimal order error estimates in the L^1 and $W^{1,1}$ norms, for adequately refined meshes; he also obtained pointwise estimates far from the singularity and the boundary. Seidman et. al. [23] consider elliptic and parabolic problems with measure-valued source terms and prove a priori estimates in L^2 and $L^p(L^2)$, respectively. A posteriori error estimates on two dimensional domains have been obtained by Araya et. al. [2, 1] for the error measured in L^p $(1 < p < \infty)$ and $W^{1,p}$ $(p_0 < p < 2)$ for certain value of p_0, and by Gaspoz et. al. [12] for the error measured in H^s $(1/2 < s < 1)$.

In a recent article, D'Angelo [6] proved the well-posedness of Poisson problem with singular sources on weighted Sobolev spaces, over three-dimensional domains, obtaining also stability and optimal estimates for a priori designed meshes, in the spirit of [9]. D'Angelo measures the error in $H^1_\alpha = H^1_{\mathrm{d}^{2\alpha}}$, where $\mathrm{d}(x) = \mathrm{dist}(x, \Lambda)$, $\alpha \in (0,1)$, and Λ is the support of the singular source term, which is a smooth curve; his results carry over immediately to two dimensional domains with point sources. The weighted Sobolev space considered by D'Angelo is "larger" than $H^1(\Omega)$ and seems to be more appropriate than the $W^{1,p}$ spaces with $p < 2$ used by Araya et. al., or the H^s spaces with $s < 1$ used by Gaspoz et. al., because the weight weakens the norm only around the singularity, letting it behave like the usual $W^{1,2} = H^1$ norm far from the location of the support of the Dirac's delta.

In this article, we develop residual type a posteriori error estimators for the norm measured in H^1_α as in [6]. We consider the following general elliptic problem on a Lipschitz domain $\Omega \subset \mathbb{R}^2$ ($\mathbb{R}^3$), with a polygonal (polyhedral) boundary $\partial\Omega$

$$\begin{cases} -\nabla \cdot \big(\mathcal{A}\nabla u\big) + \boldsymbol{b}\cdot\nabla u + cu = \delta_{x_0} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\mathcal{A} \in L^\infty(\Omega;\mathbb{R}^{n\times n})$ is piecewise $W^{1,\infty}$ and uniformly symmetric positive definite (SPD) over Ω, i.e., there exist constants $0 < \gamma_1 \le \gamma_2$ such that

$$\gamma_1|\xi|^2 \le \xi^T\mathcal{A}(x)\xi \le \gamma_2|\xi|^2, \quad \forall x \in \Omega,\ \xi \in \mathbb{R}^n, \tag{2}$$

$\boldsymbol{b} \in W^{1,\infty}(\Omega;\mathbb{R}^n)$, $c \in L^\infty(\Omega)$, and δ_{x_0} is the Dirac delta distribution supported at an inner point x_0 of Ω. We assume that $c - \frac{1}{2}\operatorname{div}(\boldsymbol{b}) \ge 0$.

The main results of this article, stated in Theorems 5.1 and 5.3, are a global upper bound for the error, measured in $H^1_\alpha(\Omega)$ for $\alpha \in \mathbb{I} \subset (\frac{n}{2}-1, \frac{n}{2})$ (see (12)), in terms of the a posteriori estimators and a local lower bound up to some oscillation term. More precisely, given a shape-regular triangulation $\mathcal{T}$, we let U be the Galerkin approximation of the exact solution u with continuous finite elements of arbitrary (fixed) degree, and prove that the *a posteriori local error estimators* η_T satisfy

$$\|U-u\|_{H^1_\alpha(\Omega)} \le C_{\mathcal{U}}\bigg(\sum_{T\in\mathcal{T}}\eta_T^2\bigg)^{1/2} \quad \text{and} \quad C_{\mathcal{L}}\,\eta_T \le \|U-u\|_{H^1_\alpha(\omega_T)} + \operatorname{osc}_T,\ \forall T \in \mathcal{T},$$

with constants $C_{\mathcal{U}}$, $C_{\mathcal{L}}$ that depend only on mesh regularity, the domain Ω, the problem coefficients and α, and can be chosen independent of α on compact subintervals of $\mathbb{I}$. The set ω_T is the patch of all neighbours of T in $\mathcal{T}$, and osc_T is an oscillation term, which is of higher order than η_T.

As we have already mentioned, a posteriori error estimates for elliptic problems with point sources have been obtained in [2, 12] for two-dimensional domains. Even though a stability result has not been proven for the norms considered in [2, 1, 12], uniqueness of discrete solutions is guaranteed by the positive definiteness of the usual stiffness matrix associated to the Laplacian. When considering the weighted spaces a discrete inf-sup condition can be proved (see Section 3), allowing us to conclude convergence of adaptive methods by resorting to the general theory developed in [17]. Moreover, our theory is valid in two and three dimensions, whereas the results from [2, 1, 12] cannot be immediately extended to the three dimensional case.

In [2] the solution is seen as an element of $W^{1,p}(\Omega)$, for some $p < 2$, and the test functions belong to $W^{1,p'}(\Omega)$, with $1/p + 1/p' = 1$ and thus $p' > 2$. By Sobolev embeddings the test functions are continuous, whence the usual proof for the upper bound can be done resorting to the Lagrange interpolant. The same happens in [12], where the solution is seen as an element of $H^{1-s}(\Omega)$ and the test functions belong to $H^{1+s}(\Omega)$ for $0 < s < 1/2$. In this article we see the solution as an element of the weighted Sobolev space $H^1_\alpha(\Omega) = \{v : \int_\Omega (v^2 + |\nabla v|^2)\,\mathrm{d}_{x_0}^{2\alpha} < \infty\}$, with $\mathrm{d}_{x_0}(x) = |x_0 - x|$ and $\frac{n}{2} - 1 < \alpha < \frac{n}{2}$, n being the dimension of the underlying domain Ω. The test functions belong to $H^1_{-\alpha}(\Omega)$, and are not necessarily continuous, but $\delta_{x_0}(v)$ is well defined for all $v \in H^1_{-\alpha}(\Omega)$. On the one hand this seems advantageous because the weight only weakens the norm around x_0, but behaves as the usual H^1 norm in subsets at a positive distance to x_0, besides the fact that estimates for two- and three-dimensional domains can be obtained. On the other hand, it does not allow us to use Lagrange interpolation because the test functions are not necessarily continuous. Instead, we resort to Clément, or Scott-Zhang operator, whose well known properties are sufficient for our purposes. In contrast to [4], where weighted spaces appear due to dimension reduction in an axisymmetrical problem, we do not need to modify the interpolation operators, but just use their local approximation and stability properties stated in (31)–(32).

The rest of this article is organized as follows. In Section 2 we define the weighted spaces and discuss the well-posedness of the problem. In Section 3 we specify the finite element spaces, and the discrete solution, proving stability of the discrete formulation. In Section 4 we prove Poincaré type and interpolation results on simplices, these will be instrumental for proving the main results in Section 5. We end the article with some numerical simulations in Section 6 illustrating the behavior of an adaptive algorithm based on the obtained a posteriori estimators.

2 Weighted spaces and weak formulation

Let $\Omega \subset \mathbb{R}^n$ be a bounded polygonal ($n = 2$) or polyhedral ($n = 3$) domain with Lipschitz boundary and x_0 an inner point of Ω. The usual test and ansatz space for elliptic problems is the Sobolev space $H_0^1(\Omega) = W_0^{1,2}(\Omega)$ when the source term belongs to its dual. The source term considered in this article does not belong to the dual space of $H_0^1(\Omega)$, because $H_0^1(\Omega)$ is not immersed into $C(\Omega)$, the space of continuous functions, but if $n = 2$ very little is missing, since $W_0^{1,p'}(\Omega)$ and $H_0^{1+s}(\Omega)$ are immersed into $C(\Omega)$ if $p' > 2$ and $s > 0$. This fact was exploited in [2] and in [12] respectively, obtaining error estimates for the error in $W_0^{1,p}(\Omega)$ and in $H_0^{1-s}(\Omega)$ for some values of $p < 2$ and $s > 0$. In three dimensions $W_0^{1,p'}(\Omega)$ and $H_0^{1+s}(\Omega)$ are immersed in $C(\Omega)$ if $p' > 3$ and $s > 3/2$, respectively, and the theory from [2, 12] cannot be extended straightforwardly.

We follow here an idea proposed in [6], where weighted spaces are used, without modifying the integrability power or the differentiability order. More precisely, for $\beta \in (-\frac{n}{2}, \frac{n}{2})$, we denote by $L^2(\Omega, \mathrm{d}_{x_0}^{2\beta})$ the space of measurable functions u such that

$$\|u\|_{L^2_\beta(\Omega)} := \|u\|_{L^2(\Omega,\mathrm{d}_{x_0}^{2\beta})} := \int_\Omega |u(x)|^2 \, \mathrm{d}_{x_0}(x)^{2\beta} dx < \infty,$$

where $\mathrm{d}_{x_0}(x) = |x - x_0|$ is the euclidean distance from x to x_0. We will write $L^2_\beta(\Omega)$ to denote $L^2(\Omega, \mathrm{d}_{x_0}^{2\beta})$ and observe that it is a Hilbert space equipped with the scalar product

$$\langle u, v\rangle_{\Omega,\beta} := \int_\Omega u(x)v(x) \, \mathrm{d}_{x_0}(x)^{2\beta} dx.$$

We also define the weighted Sobolev space $H^1_\beta(\Omega)$ of weakly differentiable functions u such that $\|u\|_{H^1_\beta(\Omega)} < \infty$, with

$$\|u\|_{H^1_\beta(\Omega)} := \|u\|_{L^2_\beta(\Omega)} + \|\nabla u\|_{L^2_\beta(\Omega)}.$$

We immediately observe that, if $0 < \alpha < \frac{n}{2}$, then $H^1_{-\alpha}(\Omega) \subset H^1(\Omega) \subset H^1_\alpha(\Omega)$ with continuity. The goal is to use appropriate subspaces of $H^1_{-\alpha}(\Omega)$ and $H^1_\alpha(\Omega)$ for the test and ansatz space, respectively. We need to prove that this leads to a stable formulation, and we thus recall some known facts about weighted spaces.

The theory of weighted L^p spaces over n-dimensional domains is well developed and much attention has been payed to the class of Muckenhoupt weights A_p, for which the Hardy-Littlewood Maximal operator is bounded in L^p. The class A_p, $1 \le p < \infty$, is defined as the sets of weights (nonnegative measurable functions) $w : \mathbb{R}^n \to \mathbb{R}_+$ that satisfy:

$$\sup_{\substack{B=B(y,r)\\ y\in\mathbb{R}^n,\, r>0}} \left(\frac{1}{|B|}\int_B w(x)\,dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1} < \infty,$$

where $B(y, r)$ is the ball centered at y with radius r, and $|B|$ is its Lebesgue measure; $A_\infty = \bigcup_{p\ge1} A_p$. The supremum on the left-hand side is called *the* A_p *constant of* w.

In our context of Hilbert spaces over two-dimensional and three-dimensional domains only the Muckenhoupt class A_2 matters, and analyzing separately the cases $|y - x_0| > 2r$ and $|y - x_0| \le 2r$ it is easy to prove that the weight function $\mathrm{d}_{x_0}^{2\beta}$ belongs to A_2 if and only if $-\frac{n}{2} < \beta < \frac{n}{2}$, because in this case

$$\frac{1}{n^2 - (2\beta)^2} \le \sup_{\substack{B=B(y,r)\\ y\in\mathbb{R}^2,\, r>0}} \left(\frac{1}{|B|}\int_B \mathrm{d}_{x_0}^{2\beta}\right)\left(\frac{1}{|B|}\int_B \mathrm{d}_{x_0}^{-2\beta}\right) \le \frac{C_n}{n^2 - (2\beta)^2},$$

for some $C_n > 1$, and the supremum is infinite if $\beta \notin (-\frac{n}{2}, \frac{n}{2})$.

If we consider $-\frac{n}{2} < \beta < \frac{n}{2}$, the results from [14, 15, 16] imply that smooth functions are dense in $H^1_\beta(\Omega)$, and also a Rellich-Kondrachov theorem and a Poincaré inequality hold in $H^1_\beta(\Omega)$.

As we will prove in Theorem 4.7, following the lines in the proof of [7, Theorem 4.2], for any $\frac{n}{2} - 1 < \alpha < \frac{n}{2}$ there exists a constant C depending on α such that

$$|\delta_{x_0}(\varphi)| = |\varphi(x_0)| \le C\|\varphi\|_{H^1_{-\alpha}(\Omega)}, \qquad \forall\varphi \in C^1(\bar{\Omega}).$$

By the density of smooth functions in $H^1_{-\alpha}(\Omega)$ we conclude that there exists a unique linear continuous map $\delta_{x_0} : H^1_{-\alpha}(\Omega) \to \mathbb{R}$ such that $\delta_{x_0}(\varphi) = \varphi(x_0)$ for any smooth function $\varphi \in C^1(\bar{\Omega})$.

Since we are considering Dirichlet boundary conditions, we define

$$W_\beta := \{u \in H^1_\beta(\Omega) : u_{|\partial\Omega} = 0\},$$

and since $\mathrm{d}_{x_0}^{2\beta}$ belongs to A_2, from [11, Theorem 1.3] it follows that Poincaré inequality holds in W_β and therefore $\|u\|_{W_\beta} := \|\nabla u\|_{L^2_\beta(\Omega)}$ is a norm in W_β equivalent to the inherited norm $\|u\|_{H^1_\beta(\Omega)}$. More precisely, there exists a constant $C_{P,\beta}$ such that

$$\|u\|_{W_\beta} \le \|u\|_{H^1_\beta(\Omega)} \le C_{P,\beta} \|u\|_{W_\beta}, \qquad u \in W_\beta, \tag{3}$$

where $C_{P,\beta}$ blows up as $|\beta|$ approaches $\frac{n}{2}$.

Given $\frac{n}{2} - 1 < \alpha < \frac{n}{2}$, the considerations above yield $W_{-\alpha} \subset H^1_0(\Omega) \subset W_\alpha$ and $\delta_{x_0} \in (W_{-\alpha})'$. We thus say that u is a weak solution of (1) if

$$u \in W_\alpha : \qquad a(u,v) = \delta_{x_0}(v), \quad \forall\, v \in W_{-\alpha}, \tag{4}$$

where $a : W_\alpha \times W_{-\alpha} \to \mathbb{R}$ is the bilinear form given by

$$a(u,v) = \int_\Omega \mathcal{A}\nabla u \cdot \nabla v + \boldsymbol{b} \cdot \nabla u\, v + c\, u\, v, \tag{5}$$

which is clearly well-defined and bounded in $W_\alpha \times W_{-\alpha}$ due to Hölder inequality.

2.1 Existence of a weak solution

We devote this section to study the well posedness of problem (4), which is a particular case of the following problem: Given $F \in (W_{-\alpha})'$,

$$\text{Find } u \in W_\alpha \quad \text{such that} \quad a(u,v) = F(v), \quad \forall\, v \in W_{-\alpha}. \tag{6}$$

Notice that u is a solution to (6) if $u = \bar{u} + \bar{w}$, with

$$\bar{u} \in W_\alpha : \qquad \int_\Omega \mathcal{A}\nabla \bar{u} \cdot \nabla v = F(v), \quad \forall\, v \in W_{-\alpha}, \qquad \text{and} \tag{7}$$

$$\bar{w} \in H^1_0(\Omega) : \qquad a(\bar{w},v) = l(v) := -\int_\Omega (\boldsymbol{b} \cdot \nabla \bar{u} + c\bar{u})v, \quad \forall\, v \in H^1_0(\Omega). \tag{8}$$

In fact, since $W_{-\alpha} \subset H^1_0(\Omega)$, from (7) and (8) we have, for $v \in W_{-\alpha}$,

$$a(u,v) = a(\bar{u},v) + a(\bar{w},v) = F(v) + \int_\Omega (\boldsymbol{b} \cdot \nabla \bar{u} + c\bar{u})v + a(\bar{w},v) = F(v).$$

Therefore, to prove the existence of solutions of (6), we just need to show that problems (7) and (8) have solutions.

Let us first consider problem (7). We will use bold symbols to denote spaces of vector valued functions, for instance, $\boldsymbol{L}^2_\beta(\Omega) = [L^2_\beta(\Omega)]^n$, and $\langle \cdot, \cdot \rangle_\Omega$ will denote both the usual $L^2(\Omega)$ and the $\boldsymbol{L}^2(\Omega)$ inner products. To prove existence and uniqueness for problem (7) we will use the following decomposition of $\boldsymbol{L}^2_\beta(\Omega)$.

Lemma 2.1 (Decomposition of $\boldsymbol{L}^2_\beta(\Omega)$)**.** *Let $\beta \in (-\frac{n}{2}, \frac{n}{2})$. For each $\boldsymbol{\tau} \in \boldsymbol{L}^2_\beta(\Omega)$, there exists a unique pair $(\boldsymbol{\sigma}, z) \in \boldsymbol{L}^2_\beta(\Omega) \times W_\beta$ such that*

$$\boldsymbol{\tau} = \nabla z + \boldsymbol{\sigma}, \quad \langle \mathcal{A}\boldsymbol{\sigma}, \nabla w \rangle_\Omega = 0 \quad \forall\, w \in W_{-\beta},$$

$$\|\nabla z\|_{L^2_\beta(\Omega)} \le 2\|\boldsymbol{\tau}\|_{L^2_\beta(\Omega)}, \quad \|\boldsymbol{\sigma}\|_{L^2_\beta(\Omega)} \le \|\boldsymbol{\tau}\|_{L^2_\beta(\Omega)}.$$

This is an immediate generalization of [6, Lemma 2.1], which states the same result for $\mathcal{A} = I$. The proof follows exactly the same lines, using that $\mathcal{A}$ is uniformly SPD over Ω, and is thus omitted.

We now use this decomposition to prove that the bilinear form of (7) satisfies an inf-sup condition. Given $u \in W_\alpha$, let $\boldsymbol{\tau} := \nabla u\, \mathrm{d}_{x_0}^{2\alpha} \in \boldsymbol{L}^2_{-\alpha}(\Omega)$. Using the decomposition of $\boldsymbol{L}^2_{-\alpha}(\Omega)$, there exist $\boldsymbol{\sigma} \in \boldsymbol{L}^2_{-\alpha}(\Omega)$ and $v \in W_{-\alpha}$ such that $\boldsymbol{\tau} = \nabla v + \boldsymbol{\sigma}$, $\langle \mathcal{A}\nabla w, \boldsymbol{\sigma}\rangle_\Omega = 0$, $\forall\, w \in W_\alpha$ and $2\,\|u\|_{W_\alpha} = 2\|\boldsymbol{\tau}\|_{\boldsymbol{L}^2_{-\alpha}(\Omega)} \geq \|v\|_{W_{-\alpha}}$. Then,

$$\langle \mathcal{A}\nabla u, \nabla v\rangle_\Omega = \langle \mathcal{A}\nabla u, \boldsymbol{\tau}\rangle_\Omega - \langle \mathcal{A}\nabla u, \boldsymbol{\sigma}\rangle_\Omega = \langle \mathcal{A}\nabla u, \nabla u\, \mathrm{d}_{x_0}^{2\alpha}\rangle_\Omega \geq \gamma_1\, \|u\|^2_{W_\alpha} \geq \frac{\gamma_1}{2}\, \|u\|_{W_\alpha}\, \|v\|_{W_{-\alpha}}\,,$$

where γ_1 is given by (2). The same estimate still holds if we swap u and v and change the sign of α. So, the following inf-sup conditions are valid:

$$\inf_{u\in W_\alpha} \sup_{v\in W_{-\alpha}} \frac{\int_\Omega \mathcal{A}\nabla u \cdot \nabla v}{\|u\|_{W_\alpha}\, \|v\|_{W_{-\alpha}}} \geq \frac{\gamma_1}{2} \quad \text{and} \quad \inf_{v\in W_{-\alpha}} \sup_{u\in W_\alpha} \frac{\int_\Omega \mathcal{A}\nabla u \cdot \nabla v}{\|u\|_{W_\alpha}\, \|v\|_{W_{-\alpha}}} \geq \frac{\gamma_1}{2}.$$

Besides, by Hölder inequality the bilinear form $A[v, w] := \int_\Omega \mathcal{A}\nabla v\cdot\nabla w$ is bounded in $W_\alpha \times W_{-\alpha}$, whence the generalized Lax-Milgram theorem due to Nečas [19, Theorem 3.3] leads to existence and uniqueness of a solution $\bar{u}$ to problem (7), which satisfies

$$\|\bar{u}\|_{W_\alpha} \leq \frac{2}{\gamma_1}\|F\|_{(W_{-\alpha})'}. \tag{9}$$

If $\boldsymbol{b} = 0$ and $c = 0$ then $\bar{u}$ is the solution to (6), and existence is proved.

Let us now consider problem (8) for $\boldsymbol{b} \neq 0$ or $c \neq 0$, given $\bar{u}$ the solution to (7). Since we have assumed $c - \frac{1}{2}\operatorname{div}(\boldsymbol{b}) \geq 0$, the bilinear form $a : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ given by (5) is continuous and coercive, and thus, by Lax-Milgram theorem, problem (8) admits a unique solution $\bar{w} \in H^1_0(\Omega)$ if $l \in H^{-1}(\Omega) := (H^1_0(\Omega))'$, where $l(v) := -\int_\Omega(\boldsymbol{b}\cdot\nabla\bar{u} + c\bar{u})v\, dx$, for $v \in H^1_0(\Omega)$. In order to prove that $l \in H^{-1}(\Omega)$ we establish first a couple of lemmas.

Lemma 2.2. *If $\alpha \in (0, \frac{n}{2})$, then for all $1 \leq p < \frac{n}{2\alpha}$, we have that $\mathrm{d}_{x_0}^{-2\alpha} \in L^p(\Omega)$.*

Proof. Let R be large enough to yield $B(x_0, R) \supset \Omega$, then

$$\int_\Omega \mathrm{d}_{x_0}^{-2\alpha p}\, dx \leq \int_{B(x_0,R)} \mathrm{d}_{x_0}^{-2\alpha p} \leq 4\pi \int_0^R r^{-2\alpha p+n-1} dr = \frac{4\pi R^{-2\alpha p+n}}{-2\alpha p + n} < \infty.$$

□

Lemma 2.3. *If $\alpha \in (0,1)$, then the following embedding holds:*

$$H^1(\Omega) \hookrightarrow L^2_{-\alpha}(\Omega).$$

Proof. Let us first consider the case $n = 2$. Given $\alpha \in (0,1)$, let p be fixed such that $1 < p < \frac{1}{\alpha}$, and denote by q its Lebesgue conjugate, that is $\frac{1}{p} + \frac{1}{q} = 1$. Then, by Hölder inequality, Lemma 2.2 and the Sobolev embedding $H^1(\Omega) \hookrightarrow L^{2q}(\Omega)$ we have that, for all $v \in H^1(\Omega)$,

$$\|v\|_{L^2_{-\alpha}(\Omega)} = \left(\int_\Omega v^2\, \mathrm{d}_{x_0}^{-2\alpha}\right)^{\frac{1}{2}} \leq \left(\int_\Omega v^{2q}\right)^{\frac{1}{2q}} \left(\int_\Omega \mathrm{d}_{x_0}^{-2\alpha p}\right)^{\frac{1}{2p}} \leq c_\alpha \|v\|_{H^1(\Omega)},$$

where c_α depends on Ω and α, and blows up when α approaches 1.

Now, assume $n = 3$, $\alpha \in (0,1)$ and $p = \frac{3}{2}$. In this case we have the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, which, together with Hölder inequality, Lemma 2.2 implies, for all $v \in H^1(\Omega)$,

$$\|v\|_{L^2_{-\alpha}(\Omega)} = \left(\int_\Omega v^2\, \mathrm{d}_{x_0}^{-2\alpha}\right)^{\frac{1}{2}} \leq \left(\int_\Omega v^6\right)^{\frac{1}{6}} \left(\int_\Omega \mathrm{d}_{x_0}^{-3\alpha}\right)^{\frac{1}{3}} \leq c_\alpha \|v\|_{H^1(\Omega)},$$

where c_α depends on Ω and α, and blows up when α approaches 1. □

As a consequence of the following proposition and (9), since $\bar{u} \in W_\alpha$, $l \in H^{-1}(\Omega)$ for $\alpha \in (\frac{n}{2} - 1, 1)$, and in fact,

$$\|l\|_{H^{-1}(\Omega)} \leq \frac{2C_{P,\alpha}c_\alpha}{\gamma_1} \max\{\|\boldsymbol{b}\|_{L^\infty(\Omega)}, \|c\|_{L^\infty(\Omega)}\}\|F\|_{(W_{-\alpha})'},$$

where $C_{P,\alpha}$ is the constant from (3) and c_α comes from Lemma 2.3

Proposition 2.4. *If $\alpha \in (\frac{n}{2} - 1, 1)$ and $\bar{u} \in W_\alpha$, then for all $v \in H^1(\Omega)$,*

$$\left| \int_\Omega (\boldsymbol{b} \cdot \nabla \bar{u} + c\bar{u})v \right| \leq C_{P,\alpha}c_\alpha \max\{\|\boldsymbol{b}\|_{L^\infty(\Omega)}, \|c\|_{L^\infty(\Omega)}\}\|\bar{u}\|_{W_\alpha}\|v\|_{H^1(\Omega)}, \tag{10}$$

with $C_{P,\alpha}$ the constant from (3) and c_α from Lemma 2.3.

Proof. From Lemma 2.3 it follows that

$$\begin{aligned}\left| \int_\Omega (\boldsymbol{b} \cdot \nabla \bar{u} + c\bar{u})v \right| &\leq \left(\|\boldsymbol{b}\|_{L^\infty(\Omega)} \|\nabla \bar{u}\|_{L^2_\alpha(\Omega)} + \|c\|_{L^\infty(\Omega)} \|\bar{u}\|_{L^2_\alpha(\Omega)} \right) \|v\|_{L^2_{-\alpha}(\Omega)} \\ &\leq \max\{\|\boldsymbol{b}\|_{L^\infty(\Omega)}, \|c\|_{L^\infty(\Omega)}\}(\|\bar{u}\|_{L^2_\alpha(\Omega)} + \|\nabla \bar{u}\|_{L^2_\alpha(\Omega)})c_\alpha\|v\|_{H^1(\Omega)},\end{aligned}$$

and the assertion follows from (3). □

Therefore, for $\alpha \in (\frac{n}{2} - 1, 1)$, by Lax-Milgram theorem we conclude that problem (8) admits a unique solution $\bar{w}$ satisfying

$$\|\bar{w}\|_{H^1_0(\Omega)} \leq CC_{P,\alpha}c_\alpha\|F\|_{(W_{-\alpha})'}, \tag{11}$$

where C is a constant depending on the problem coefficients.

From now on, we will consider

$$\alpha \in \mathbb{I} := \begin{cases} (0, 1) & \text{if} \quad n = 2 \quad \text{and} \quad \boldsymbol{b} \in W^{1,\infty}(\Omega; \mathbb{R}^2),\ c \in L^\infty(\Omega), \\ (\frac{1}{2}, 1) & \text{if} \quad n = 3 \quad \text{and} \quad \boldsymbol{b} \in W^{1,\infty}(\Omega; \mathbb{R}^3),\ c \in L^\infty(\Omega), \\ (\frac{1}{2}, \frac{3}{2}) & \text{if} \quad n = 3 \quad \text{and} \quad \boldsymbol{b} = 0,\ c = 0, \end{cases} \tag{12}$$

and summarize the results of this section as follows. If $\bar{u}$ denotes the unique solution of problem (7) and $\bar{w}$ denotes the unique solution of problem (8) then $u := \bar{u} + \bar{w}$ is a solution of problem (6), and from (9) and (11), we get

$$\|u\|_{W_\alpha} \leq C_*\|F\|_{(W_{-\alpha})'}, \tag{13}$$

with the constant C_* depending on the domain Ω, the problem coefficients and α, and blows up when α approaches the right endpoint of $\mathbb{I}$, except when $\boldsymbol{b} = 0$ and $c = 0$, because in this case $C_* = 2/\gamma_1$, which is independent of α.

2.2 Uniqueness of weak solution

In this section we prove uniqueness of solution $u \in W_\alpha$ of (6) for $\alpha \in \mathbb{I}$.

If $\boldsymbol{b} = 0$ and $c = 0$, (6) coincides with (7) and uniqueness is a consequence of (9).

If $\boldsymbol{b} \neq 0$ or $c \neq 0$, suppose that $\tilde{u}$ is a solution of (6). We define the function $\tilde{w} := \tilde{u} - \bar{u}$, with $\bar{u}$ the unique solution of (7). Then,

$$\tilde{w} \in W_\alpha : \qquad a(\tilde{w}, v) = -\int_\Omega (\boldsymbol{b} \cdot \nabla \bar{u} + c\bar{u})v, \quad \forall v \in W_{-\alpha}. \tag{14}$$

Now, we consider the following problem:

$$\text{Find } w \in W_\alpha \text{ such that} \quad \int_\Omega \mathcal{A}\nabla w \cdot \nabla v = L(v), \qquad \forall v \in W_{-\alpha}, \tag{15}$$

where L is given by

$$L(v) := -\int_\Omega (\boldsymbol{b} \cdot \nabla \tilde{u} + c\tilde{u})v.$$

Since $\tilde{u} \in W_\alpha$, we have that $L \in (W_{-\alpha})'$. Therefore problem (15) admits a unique solution w and we thus conclude from (14) that $w = \tilde{w}$.

Consider now the problem:

$$\text{Find } w_0 \in H_0^1(\Omega) \text{ such that } \quad \int_\Omega \mathcal{A}\nabla w_0 \cdot \nabla v = L(v), \qquad \forall\, v \in H_0^1(\Omega).$$

By Lemma 2.3, we have that $L \in H^{-1}(\Omega)$ and as a consequence this problem admits a unique solution w_0 in $H_0^1(\Omega)$. Since $W_{-\alpha} \subset H_0^1(\Omega) \subset W_\alpha$, w_0 is also a solution of problem (15) and thus $w_0 = w = \tilde{w} \in H_0^1(\Omega)$. Hence, $\tilde{w}$ is a solution of problem (8) and by the uniqueness of this problem we have that $\tilde{w} = \bar{w}$ or equivalently $\tilde{u} = \bar{u} + \bar{w}$. Therefore, $\tilde{u} = u$ and the solution of problem (6) is unique.

Finally, existence and uniqueness of solution to problem (6) for each $F \in (W_{-\alpha})'$ and the bound (13) imply that the following inf-sup condition holds (cf. [19, Theorem 3.3]):

$$\inf_{u \in W_\alpha} \sup_{v \in W_{-\alpha}} \frac{a(u,v)}{\|u\|_{W_\alpha} \|v\|_{W_{-\alpha}}} = \frac{1}{C_*}. \tag{16}$$

Note that the inf-sup constant $\frac{1}{C_*}$ depends on problem data and α, and degenerates toward 0 when α approaches the right endpoint of $\mathbb{I}$, except when $\boldsymbol{b} = 0$ and $c = 0$, because in this case $C_* = 2/\gamma_1$, which is independent of α.

3 Finite element discretization

In this section we define the finite element spaces that we consider, and let the discrete solution U be the usual Galerkin approximation of the weak solution u. We then show that the discretization is stable by proving an inf-sup condition which is independent of the mesh, which can be graded, but must be shape-regular.

3.1 Discrete setting

Let $\mathcal{T}$ be a conforming triangulation of the domain $\Omega \subset \mathbb{R}^n$. That is, a partition of Ω into n-simplices such that if two elements intersect, they do so at a full vertex/edge/face of both elements. We define the mesh regularity constant

$$\kappa := \sup_{T \in \mathcal{T}} \frac{\operatorname{diam}(T)}{\rho_T},$$

where $\operatorname{diam}(T)$ is the diameter of T, and ρ_T is the radius of the largest ball contained in it. Also, the diameter of any element $T \in \mathcal{T}$ is equivalent to the local mesh-size $h_T := |T|^{1/n}$, with equivalent constants depending on κ.

On the other hand, we denote the subset of $\mathcal{T}$ consisting of an element T and its neighbors by $\mathcal{N}_T$ and the union of the elements in $\mathcal{N}_T$ by ω_T. More precisely, for $T \in \mathcal{T}$,

$$\mathcal{N}_T := \{T' \in \mathcal{T} \mid\ T \cap T' \neq \emptyset\}, \qquad \omega_T := \bigcup_{T' \in \mathcal{N}_T} T'.$$

We denote by $\mathcal{E}_\Omega$ to the set of sides (edges for $n = 2$ and faces for $n = 3$) of the elements in $\mathcal{T}$ which are inside Ω and by $\mathcal{E}_{\partial\Omega}$ to the set of sides which lie on the boundary of Ω. We define ω_S as the union of the two elements sharing S, if $S \in \mathcal{E}_\Omega$, and as the unique element T_S satisfying $S \subset \partial T_S$ if $S \in \mathcal{E}_{\partial\Omega}$.

For the discretization we consider Lagrange finite elements of degree $\ell \in \mathbb{N}$, more precisely, we let

$$\mathbb{V}_{\mathcal{T}}^\ell := \{V \in H_0^1(\Omega) \mid\ V_{|_T} \in \mathcal{P}_\ell(T),\ \forall\, T \in \mathcal{T}\},$$

and observe that $\mathbb{V}_{\mathcal{T}}^\ell \subset W_\beta$, for $\beta \in (-\frac{n}{2}, \frac{n}{2})$. The discrete counterpart of (4) reads:

$$\text{Find } U \in \mathbb{V}_{\mathcal{T}}^\ell \text{ such that } \quad a(U,V) = \delta_{x_0}(V), \quad \forall\, V \in \mathbb{V}_{\mathcal{T}}^\ell. \tag{17}$$

Clearly, this discrete problem has a unique solution for each mesh; the system matrix is not affected by the right-hand side and is invertible because the assumptions on the problem coefficients guarantee the coercivity of the bilinear form $a(\cdot,\cdot)$ in $\mathbb{V}_{\mathcal{T}}^\ell \times \mathbb{V}_{\mathcal{T}}^\ell$.

Unlike [2, 1, 12] we also prove here a stability result, which by the theory of [17] will allow us to conclude that adaptive algorithms with the a posteriori estimates developed here yield convergence. Recall also that the discrete inf-sup is usually not used for the derivation of a posteriori estimates, only the continuous one needs to be used.

3.2 Stability of the discrete problems

As we did for the infinite-dimensional problem, note that problem (17) is a particular case of the following problem for a fixed $\frac{n}{2}-1<\alpha<\frac{n}{2}$: Given $F\in(W_{-\alpha})'$, find

$$U\in\mathbb{V}_{\mathcal{T}}^{\ell}: \qquad a(U,V)=F(V), \qquad \forall\, V\in\mathbb{V}_{\mathcal{T}}^{\ell}. \tag{18}$$

We also split $U=\bar{U}+\bar{W}$ with

$$\bar{U}\in\mathbb{V}_{\mathcal{T}}^{\ell}: \qquad \int_\Omega \mathcal{A}\nabla\bar{U}\cdot\nabla V=F(V), \quad \forall\, V\in\mathbb{V}_{\mathcal{T}}^{\ell}, \tag{19}$$

$$\bar{W}\in\mathbb{V}_{\mathcal{T}}^{\ell}: \qquad a(\bar{W},V)=-\int_\Omega(\boldsymbol{b}\cdot\nabla\bar{U}+c\bar{U})V, \quad \forall\, V\in\mathbb{V}_{\mathcal{T}}^{\ell}. \tag{20}$$

Then, defining $U=\bar{U}+\bar{W}$, we have

$$a(U,V)=a(\bar{U},V)+a(\bar{W},V)=F(V)+\int_\Omega(\boldsymbol{b}\cdot\nabla\bar{U}+c\bar{U})V+a(\bar{W},V)=F(V),$$

for all $V\in\mathbb{V}_{\mathcal{T}}^{\ell}$. That is, U is a solution to (18). Therefore, we just need to bound the solutions to problems (19) and (20) in W_α by $\|F\|_{(W_{-\alpha})'}$.

To do so, we apply a discrete decomposition lemma to the space

$$\mathcal{M}_{\mathcal{T}}^{\ell-1}:=\{\boldsymbol{\lambda}\in\boldsymbol{L}^2(\Omega)\mid\ \boldsymbol{\lambda}_{|_T}\in\mathcal{P}_{\ell-1}^n(T),\ \forall\, T\in\mathcal{T}\}\supset\nabla\mathbb{V}_{\mathcal{T}}^{\ell}.$$

The bounds in the decomposition are obtained for the following discrete norm in $\mathcal{M}_{\mathcal{T}}^{\ell-1}$ which is equivalent to the $L^2_\beta(\Omega)$ norm, for $\beta\in(-\frac{n}{2},\frac{n}{2})$:

$$\|\boldsymbol{\lambda}\|_{\mathcal{T},\beta}:=\left(\sum_{T\in\mathcal{T}}D_T^{2\beta}\|\boldsymbol{\lambda}\|^2_{L^2(T)}\right)^{\frac{1}{2}}, \qquad \forall\,\boldsymbol{\lambda}\in\mathcal{M}_{\mathcal{T}}^{\ell-1},$$

where $D_T:=\max_{x\in T}\mathrm{d}_{x_0}(x)$. D'Angelo proposed this discrete norm and proved in [6, Lemma 3.2] that it is equivalent to $\|\boldsymbol{\lambda}\|_{L^2_\beta(\Omega)}$, with equivalence depending only on κ, the polynomial degree ℓ and $|\beta|$. The proof is based on the fact that for $t\in(0,\frac{n}{2})$ fixed, there exists a constant c_t, depending on κ, ℓ and t, such that, if $|\beta|\le t$, then

$$\frac{1}{c_t}\|V\|_{L^2_\beta(T)}\le D_T^\beta\|V\|_{L^2(T)}\le c_t\|V\|_{L^2_\beta(T)}, \quad \forall\, T\in\mathcal{T},\ \forall\, V\in\mathcal{P}_\ell(T). \tag{21}$$

The following lemma is an immediate generalization of [6, Lemma 3.3], with a similar proof, again, taking into account that $\mathcal{A}$ is uniformly SPD.

Lemma 3.1 (Decomposition of $\mathcal{M}_{\mathcal{T}}^{\ell-1}$)**.** *Let $\beta\in(-\frac{n}{2},\frac{n}{2})$. For each $\boldsymbol{\lambda}\in\mathcal{M}_{\mathcal{T}}^{\ell-1}$, there exists a unique couple $(\boldsymbol{\sigma},Z)\in\mathcal{M}_{\mathcal{T}}^{\ell-1}\times\mathbb{V}_{\mathcal{T}}^{\ell}$ such that*

$$\boldsymbol{\lambda}=\nabla Z+\boldsymbol{\sigma}, \quad \langle\mathcal{A}\boldsymbol{\sigma},\nabla W\rangle_\Omega=0 \quad \forall\, W\in\mathbb{V}_{\mathcal{T}}^{\ell},$$

$$\|\nabla Z\|_{\mathcal{T},\beta}\le 2\|\boldsymbol{\lambda}\|_{\mathcal{T},\beta}, \quad \|\boldsymbol{\sigma}\|_{\mathcal{T},\beta}\le\|\boldsymbol{\lambda}\|_{\mathcal{T},\beta}.$$

By the same kind of arguments as in the continuous case we arrive at

$$\inf_{U\in\mathbb{V}_{\mathcal{T}}^{\ell}}\sup_{V\in\mathbb{V}_{\mathcal{T}}^{\ell}}\frac{\int_\Omega\mathcal{A}\nabla U\cdot\nabla V}{\|U\|_{W_\alpha}\|V\|_{W_{-\alpha}}}\ge\tilde{\gamma}_1 \quad\text{and}\quad \inf_{V\in\mathbb{V}_{\mathcal{T}}^{\ell}}\sup_{U\in\mathbb{V}_{\mathcal{T}}^{\ell}}\frac{\int_\Omega\mathcal{A}\nabla U\cdot\nabla V}{\|U\|_{W_\alpha}\|V\|_{W_{-\alpha}}}\ge\tilde{\gamma}_1,$$

where $\tilde{\gamma}_1$ depends on γ_1 from (2), κ, ℓ and α, and goes to zero when α approaches $\frac{n}{2}$. The unique solution $\bar{U}$ of problem (19) satisfies

$$\|\bar{U}\|_{W_\alpha} \leq \frac{1}{\tilde{\gamma}_1}\|F\|_{(W_{-\alpha})'}. \tag{22}$$

On the other hand, in view of Proposition 2.4, for $\alpha \in \mathbb{I}$, the linear form

$$\bar{L}(V) := -\int_\Omega (\boldsymbol{b}\cdot\nabla\bar{U} + c\bar{U})V$$

satisfies $\|\bar{L}\|_{H^{-1}(\Omega)} \leq C\|\bar{U}\|_{W_\alpha}$. Since the continuity and coercivity of the bilinear form a is inherited from the continuous space to the discrete one, the solution $\bar{W}$ of problem (20) satisfies

$$\|\bar{W}\|_{H^1_0(\Omega)} \leq C\|\bar{U}\|_{W_\alpha}, \tag{23}$$

where the generic constant C is independent of $\mathcal{T}$.

Finally, taking into account (22) and (23), we get

$$\|U\|_{W_\alpha} \leq C\|F\|_{(W_{-\alpha})'},$$

where the generic constant C depends on $\mathcal{T}$ solely through κ, problem data, the polynomial degree ℓ, the parameter $\alpha \in \mathbb{I}$, and blows up as α approaches the right endpoint of $\mathbb{I}$.

4 Some results in weighted spaces on simplices

In this section we state and prove some properly scaled bounds which are valid on the elements of the triangulation, with constants depending only on mesh regularity. These bounds include a local Poincaré inequality, a bound for $\|\delta_{x_0}\|_{(W_{-\alpha})'}$, and bounds for the error in Clément and Scott-Zhang interpolation operators. Most of these bounds are known for the usual Sobolev norms, without weights.

This section is independent of the elliptic operator or the precise problem at hand. The results stated here might be useful in other applications involving point sources.

From now on, we will write $a \lesssim b$ to indicate that $a \leq Cb$ with $C > 0$ a constant depending on the shape regularity κ of the mesh and possibly on the domain $\Omega \subset \mathbb{R}^n$, which is assumed polygonal ($n = 2$) or polyhedral ($n = 3$) with a Lipschitz boundary. Also $a \simeq b$ will indicate that $a \lesssim b$ and $b \lesssim a$.

4.1 Classification of simplices

In order to prove our results we classify the elements according to their relationship to x_0. We categorize the elements of $\mathcal{T}$ into two disjoint classes, defined as follows:

$$\mathcal{T}^{\text{near}} := \{T \in \mathcal{T} \mid x_0 \in \omega_T\} \qquad \text{and} \qquad \mathcal{T}^{\text{far}} := \mathcal{T} \setminus \mathcal{T}^{\text{near}}.$$

We establish a relationship between the classical local norms $\|\cdot\|_{L^2(T)}$ and the weighted ones $\|\cdot\|_{L^2_\beta(T)}$.

Lemma 4.1. *The following statements hold:*

(i) If $-\frac{n}{2} < \beta < \frac{n}{2}$ and $T \in \mathcal{T}^{\text{far}}$, then $h_T \lesssim d_T \simeq D_T$ and

$$\|v\|_{L^2_\beta(T)} \simeq D_T^\beta \|v\|_{L^2(T)}, \qquad \forall\, v \in L^2(T), \tag{24}$$

$$\|v\|_{L^2_\beta(\partial T)} \simeq D_T^\beta \|v\|_{L^2(\partial T)}, \qquad \forall\, v \in L^2(\partial T). \tag{25}$$

(ii) If $0 \leq \alpha < \frac{n}{2}$ and $T \in \mathcal{T}^{\text{near}}$, then $h_T \simeq D_T$ and

$$\|v\|_{L^2_{-\alpha}(T)} \gtrsim h_T^{-\alpha}\|v\|_{L^2(T)}, \qquad \forall\, v \in L^2_{-\alpha}(T), \tag{26}$$

$$\|v\|_{L^2_\alpha(T)} \lesssim h_T^{\alpha}\|v\|_{L^2(T)}, \qquad \forall\, v \in L^2(T). \tag{27}$$

Proof. Let $T \in \mathcal{T}^{\text{far}}$, then $x_0 \notin \omega_T$, and $d_T = \min_T \mathrm{d}_{x_0} = |x_0 - x|$ for some $x \in T$, whence $h_T \lesssim d_T$ by Lemma 4.2 below. Therefore, $D_T \lesssim d_T + h_T \lesssim d_T$ and thus $d_T \simeq D_T$, which implies (24). Since $d_T \le \min_{\partial T} \mathrm{d}_{x_0} \le \max_{\partial T} \mathrm{d}_{x_0} \le D_T$, (25) holds.

Let $T \in \mathcal{T}^{\text{near}}$. Then $x_0 \in \omega_T$, and thus $D_T \le \operatorname{diam}(\omega_T) \lesssim h_T$. Besides, if x_1, x_2 are two vertices of T,

$$h_T \simeq |x_1 - x_2| \le |x_1 - x_0| + |x_0 - x_2| \le 2D_T.$$

Therefore $h_T \simeq D_T$, and thus (26) and (27) hold. $\square$

The following lemma states that a neigborhood of size $\simeq h_T$ of an element T is always contained in ω_T. This is known in the finite element community, but we could not find a proof. Since the one we found is very short we decided to include it here for completeness. This result was used in the previous lemma, and will be used in the proof of the lower bound (see Theorem 5.3).

Lemma 4.2. *There exists a constant $c_{\kappa,\Omega} > 0$ depending on mesh regularity κ and the Lipschitz property of $\partial\Omega$ such that, if $T \in \mathcal{T}$, $x \in T$ and $y \in \Omega\backslash\omega_T$, then $|x-y| \ge c_{\kappa,\Omega} h_T$. In other words, $B(x, c_{\kappa,\Omega} h_T) \cap \Omega \subset \omega_T$ for all $x \in T$ and all $T \in \mathcal{T}$.*

Proof. Let $T \in \mathcal{T}$, let ϕ_i, $i = 1, \dots, n+1$, be the canonical basis functions of $\mathbb{V}^1_{\mathcal{T}}$ corresponding to each vertex of T, and let $\psi = \sum_{i=1}^{n+1} \phi_i$. Then $\|\nabla\psi\|_{L^\infty(\Omega)} \lesssim 1/h_T$, and therefore

$$|\psi(x) - \psi(y)| \le \frac{1}{c_{\kappa,\Omega} h_T} |x - y|, \qquad \text{for all } x, y \in \Omega,$$

where $c_{\kappa,\Omega}$ depends only on mesh regularity and the Lipschitz property of $\partial\Omega$. Since $\psi(x) = 1$ if $x \in T$ and $\psi(y) = 0$ for $y \notin \omega_T$ the claim follows. $\square$

4.2 Local Poincaré inequality and interpolation estimates

The usual scaling arguments used to prove Poincaré inequalities on simplices do not lead to a uniform constant for all the elements in the mesh. We thus need to resort to real analysis tools from the theory of weighted inequalities [18, 11]. We start by recalling some definitions and important properties.

Let $0 < \gamma < n$, the fractional integral $T_\gamma(f)$ and the fractional maximal function f^*_γ of a measurable function $f : \mathbb{R}^n \to \mathbb{R}$ are defined, for $x \in \mathbb{R}^n$ by

$$T_\gamma(f)(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}}\, dy, \qquad f^*_\gamma(x) := \sup_B \frac{1}{|B|^{1-\gamma/n}} \int_B |f(y)|\, dy, \tag{28}$$

where the supremum is taken over all balls B with center at x.

These two concepts are related through the following result, proved by Muckenhoupt and Wheeden (cf. [18, Theorem 1]), for any $n \in \mathbb{N}$.

Lemma 4.3. *Let $0 < \gamma < n$, $w \in A_\infty = \cup_{q \ge 1} A_q$, and $1 < p < \infty$. Then, there exists a constant $c > 0$ such that*

$$\left(\int_{\mathbb{R}^n} |T_\gamma(f)|^p w \right)^{\frac{1}{p}} \le c \left(\int_{\mathbb{R}^n} |f^*_\gamma|^p w \right)^{\frac{1}{p}},$$

for all measurable functions f.

From [11, Lemma 1.1] and using the same arguments of the proof of [11, Theorem 1.2] the next result follows, for the particular case $\gamma = 1$.

Lemma 4.4. *Let $w \in A_p$, for some p, $1 < p < \infty$. Then, there exists a constant $c > 0$, depending only on the A_p constant of w, such that*

$$\left(\int_{\mathbb{R}^n} |f^*_1|^p w \right)^{\frac{1}{p}} \le cR \left(\int_{B_R} |f|^p w \right)^{\frac{1}{p}},$$

for all ball B_R of radius $R > 0$, and for all f measurable and supported in B_R.

As a consequence of these results we obtain the following scaled Poincaré inequality.

Theorem 4.5 (Poincaré inequality)**.** *Let $\beta \in (-\frac{n}{2}, \frac{n}{2})$. There exists a constant $C_P > 0$ depending on β and the mesh regularity κ such that, for all $v \in H^1_\beta(\Omega)$,*

$$\|v - v_T\|_{L^2_\beta(T)} \le C_P h_T \|\nabla v\|_{L^2_\beta(T)}, \qquad \forall T \in \mathcal{T},$$

where $v_T := \frac{1}{|T|}\int_T v$. The constant C_P blows up when $|\beta|$ approaches $\frac{n}{2}$.

As we mentioned earlier, the usual scaling arguments do not yield a uniform constant C_P, and we thus resort to arguments from [11], where weighted Poincaré inequalities are proved on balls, with a uniform constant depending only on the A_p constant of the weight.

Proof. Let $v \in C^1(\bar{\Omega})$ and $T \in \mathcal{T}$. Since T is convex, by [13, Lemma 7.16, p. 162] we have that

$$|v(x) - v_T| \le \frac{\operatorname{diam}(T)^n}{n\, h_T^n} \int_T \frac{|\nabla v(z)|}{|x-z|^{n-1}}\, dz,$$

for every $x \in T$. Let B_R be a ball containing T such that $R \lesssim h_T$, and define $f := |\nabla v|\chi_T$, where χ_T is the characteristic function of T. Then recalling the definition (28), $\int_T \frac{|\nabla v(z)|}{|x-z|^{n-1}}\, dz = T_1(f)(x)$ and thus by mesh regularity

$$|v(x) - v_T| \lesssim T_1(f)(x), \qquad \text{a.e. } x \in T. \tag{29}$$

Since $\mathrm{d}_{x_0}^{2\beta} \in A_2 \subset A_\infty$, due to Lemmas 4.3 and 4.4 it follows that

$$\|T_1(f)\|_{L^2_\beta(\mathbb{R}^n)} \le cR\|f\|_{L^2_\beta(B_R)} = cR\|\nabla v\|_{L^2_\beta(T)}, \tag{30}$$

for some constant $c > 0$, depending only on β, through the A_2 constant of $\mathrm{d}_{x_0}^{2\beta}$, which blows up as $|\beta|$ approaches $n/2$. The bounds (29) and (30) yield the result for smooth functions v. The assertion of the theorem follows by density arguments. □

We will now show some interpolation estimates in weighted spaces, which hinge on the Poincaré inequality from Theorem 4.5, and are instrumental for proving the reliability of the error estimators. Let $\mathcal{P} : H^1_0(\Omega) \to \mathbb{V}^1_{\mathcal{T}}$ be either the Clément or the Scott-Zhang interpolation operator. It is well known [5, 22] that, for all $v \in H^1(\Omega)$,

$$\|v - \mathcal{P}v\|_{L^2(T)} \lesssim h_T \|\nabla v\|_{L^2(\omega_T)}, \quad \forall T \in \mathcal{T}, \tag{31}$$

$$\|\nabla(v - \mathcal{P}v)\|_{L^2(T)} \lesssim \|\nabla v\|_{L^2(\omega_T)}, \quad \forall T \in \mathcal{T}. \tag{32}$$

Since $H^1_{-\alpha}(\Omega) \subset H^1(\Omega)$ for $\alpha > 0$, $\mathcal{P}$ is also well defined for functions in $H^1_{-\alpha}(\Omega)$. Moreover, the above estimates hold in weighted norms, as we show in the following proposition.

Proposition 4.6 (Interpolation estimates)**.** *Let $\mathcal{P}$ denote either the Clément or the Scott-Zhang interpolation operator. Let $t \in (0, \frac{n}{2})$ and $0 \le \alpha \le t$. Then, there exists a constant $C_I > 0$ depending on the mesh regularity κ and t such that, for all $v \in H^1_{-\alpha}(\Omega)$,*

$$\|v - \mathcal{P}v\|_{L^2_{-\alpha}(T)} \le C_I h_T \|\nabla v\|_{L^2_{-\alpha}(\omega_T)}, \quad \forall T \in \mathcal{T}, \tag{33}$$

$$\|\nabla(v - \mathcal{P}v)\|_{L^2_{-\alpha}(T)} \le C_I \|\nabla v\|_{L^2_{-\alpha}(\omega_T)}, \quad \forall T \in \mathcal{T}. \tag{34}$$

The constant C_I blows up as t approaches $n/2$.

Proof. Let $v \in H^1_{-\alpha}(\Omega)$. Let $T \in \mathcal{T}$ and $v_T := \frac{1}{|T|}\int_T v$. Then, by (21)

$$\begin{aligned}
\|v - \mathcal{P}v\|_{L^2_{-\alpha}(T)} &\le \|v - v_T\|_{L^2_{-\alpha}(T)} + c_t D_T^{-\alpha} \|v_T - \mathcal{P}v\|_{L^2(T)} \\
&\le \|v - v_T\|_{L^2_{-\alpha}(T)} + c_t D_T^{-\alpha} \big(\|v_T - v\|_{L^2(T)} + \|v - \mathcal{P}v\|_{L^2(T)} \big) \\
&\lesssim \|v - v_T\|_{L^2_{-\alpha}(T)} + c_t h_T D_T^{-\alpha} \|\nabla v\|_{L^2(\omega_T)},
\end{aligned}$$

where the last inequality follows from the classic Poincaré inequality and (31). From Theorem 4.5 and the fact that $\mathrm{d}_{x_0}(x) \lesssim D_T$ for all $x \in \omega_T$ (33) holds.

Observe now that due to (21) and (32),

$$\|\nabla \mathcal{P}v\|_{L^2_{-\alpha}(T)} \le c_t D_T^{-\alpha} \|\nabla \mathcal{P}v\|_{L^2(T)} \lesssim c_t D_T^{-\alpha} \|\nabla v\|_{L^2(\omega_T)} \lesssim c_t \|\nabla v\|_{L^2_{-\alpha}(\omega_T)},$$

where we have used again that $\mathrm{d}_{x_0} \lesssim D_T$ in ω_T. The assertion (34) follows. □

4.3 A local bound for δ_{x_0}

In this section we present a local bound for δ_{x_0}, which is useful to establish the reliability of the a posteriori error estimators (cf. Theorem 5.1 below). It is a local version of [7, Theorem 4.2], and as a consequence of this result, if $\frac{n}{2}-1<\alpha<\frac{n}{2}$, there is a unique linear continuous map

$$\delta_{x_0} : H^1_{-\alpha}(\Omega) \to \mathbb{R}$$

such that $\delta_{x_0}(\varphi)=\varphi(x_0)$ for each smooth function $\varphi \in C^\infty(\Omega)$. Our proof follows the same lines, but we include it here to show the precise dependence on the scaling parameter.

Theorem 4.7 (A precise bound of δ_{x_0}). *Let $\frac{n}{2}-1<\alpha<\frac{n}{2}$ and $T\in\mathcal{T}$ such that $x_0\in T$. Then*

$$|\delta_{x_0}(v)| \lesssim h_T^{\alpha-\frac{n}{2}} \|v\|_{L^2_{-\alpha}(T)} + C_\alpha h_T^{\alpha+\frac{2-n}{2}} \|\nabla v\|_{L^2_{-\alpha}(T)}, \qquad \forall\, v\in H^1_{-\alpha}(T), \tag{35}$$

where $C_\alpha := \frac{\alpha^{\frac{\alpha-1}{2}}}{(\alpha+1)^{\frac{\alpha+1}{2}}}$ if $n=2$ and $C_\alpha := \frac{(2\alpha-1)^{\frac{\alpha-2}{3}}}{(2\alpha+2)^{\frac{\alpha+1}{3}}}$ if $n=3$.

Note that the constant C_α blows up as α approaches $\frac{n}{2}-1$. This was expected because δ_{x_0} does not belong to the dual space of $H^1_{-\alpha}(\Omega)$, for $\alpha=\frac{n}{2}-1$, but only for $\frac{n}{2}-1<\alpha<\frac{n}{2}$.

Proof. Assume $n=3$ and let $T\in\mathcal{T}$ such that $x_0\in T$. By mesh regularity, there exist constants θ_0, θ_1, ϕ_0, ϕ_1 and c_0, depending only on κ, such that a sector S_T with center at x_0 described in local spherical coordinates by

$$\{(r,\theta,\phi) \mid 0\le r\le c_0h_T,\ \theta_0\le\theta\le\theta_1,\ \phi_0\le\phi\le\phi_1\},$$

is contained in T. Let $\varphi\in C^1(T)$. Then, by using local spherical coordinates centered at x_0 we have for every $r\in(0,c_0h_T)$, $\theta\in(\theta_0,\theta_1)$ and $\phi\in(\phi_0,\phi_1)$,

$$\varphi(0,0,0)=\varphi(r,\theta,\phi)-\int_0^r \frac{\partial\varphi}{\partial r}(t,\theta,\phi)\,dt,$$

so that, using the inequality $(a+b)^2\le 2a^2+2b^2$, and integrating on S_T we get

$$\begin{aligned} Ch_T^3\varphi(0,0,0)^2 \le& \int_{\phi_0}^{\phi_1}\int_{\theta_0}^{\theta_1}\int_0^{c_0h_T} \varphi(r,\theta,\phi)^2 r^2\sin(\theta)drd\theta d\phi \\ &+\int_{\phi_0}^{\phi_1}\int_{\theta_0}^{\theta_1}\int_0^{c_0h_T} \left(\int_0^r \frac{\partial\varphi}{\partial r}(t,\theta,\phi)dt\right)^2 r^2\sin(\theta)drd\theta d\phi, \end{aligned}$$

where $C=\frac{(\phi_1-\phi_0)(cos(\theta_0)-cos(\theta_1))c_0^3}{6}$. To bound the second term we will use the weighted Hardy inequality (see Theorem 4.8 below) with $p=q=2$, the weight functions being $w_1(t)=t^2$, $w_2(t)=t^{2-2\alpha}$ and the positive function $f(t)=|\partial\varphi/\partial r(t,\theta)|$. Since $\alpha>\frac{1}{2}$, we have

$$\int_0^r w_2(t)^{\frac{1}{1-p}}dt=\int_0^r t^{2\alpha-2}dt=\frac{r^{2\alpha-1}}{2\alpha-1}<\infty, \qquad \forall\, r>0,$$

and

$$\begin{aligned} D_\alpha :=& \sup_{r\in(0,c_0h_T)} \left(\int_r^{c_0h_T} t^2\,dt\right)^{\frac{1}{2}} \left(\int_0^r t^{2\alpha-1}\,dt\right)^{\frac{1}{2}} \\ =& \sup_{r\in(0,c_0h_T)} \left[\frac{(c_0h_T)^3-r^3}{3}\frac{r^{2\alpha-1}}{2\alpha-1}\right]^{\frac{1}{2}} = h_T^{1+\alpha}\frac{c_0^{1+\alpha}(2\alpha-1)^{\frac{\alpha-2}{3}}}{(2\alpha+2)^{\frac{\alpha+1}{3}}}<\infty. \end{aligned}$$

Hence, by Theorem 4.8, the following inequality is valid

$$\int_0^{c_0h_T}\left(\int_0^r \frac{\partial\varphi}{\partial r}(t,\theta,\phi)dt\right)^2 r^2dr \le 4D_\alpha^2\int_0^{c_0h_T}\left|\frac{\partial\varphi}{\partial r}(t,\theta,\phi)\right|^2 r^{2-2\alpha}dr.$$

Therefore, using the identity $r = \mathrm{d}_{x_0}(x)$ in S_T and $1 \le \mathrm{d}_{x_0}(x)^{-2\alpha}(c_0 h_T)^{2\alpha}$, for all $x \in S_T$, we obtain

$$Ch_T^3 \varphi(0,0,0)^2 \le c_0^{2\alpha} h_T^{2\alpha} \|\varphi\|^2_{L^2_{-\alpha}(T)} + 4D_\alpha^2 \|\nabla\varphi\|^2_{L^2_{-\alpha}(T)},$$

and thus

$$|\varphi(0,0,0)| \lesssim h_T^{\alpha-\frac{3}{2}} \|\varphi\|_{L^2_{-\alpha}(T)} + C_\alpha h_T^{\alpha-\frac{1}{2}} \|\nabla\varphi\|_{L^2_{-\alpha}(T)},$$

where $C_\alpha = \frac{(2\alpha-1)^{\frac{\alpha-2}{3}}}{(2+2\alpha)^{\frac{\alpha+1}{3}}}$. The assertion follows by the density of $C^1(T)$ in $H^1_{-\alpha}(T)$.

For the case $n = 2$, the proof follows the same lines, considering a circular sector described by polar coordinates inside the triangle and the weight functions being $w_1(t) = t$, $w_2(t) = t^{1-2\alpha}$. $\square$

We end this section by stating a Hardy inequality [20] that was used in the proof of the previous result.

Theorem 4.8 (Weighted Hardy inequality)**.** *Let $0 < p \le q < \infty$, $0 < R \le \infty$ and w_1 and w_2 be weight functions defined on $(0,\infty)$. Assume that, for every $r > 0$,*

$$\int_0^r w_2(t)^{\frac{1}{1-p}}\,dt < \infty.$$

Then, the inequality

$$\left(\int_0^R \left(\int_0^r f(t)dt\right)^q w_1(r)dr\right)^{\frac{1}{q}} \le C\left(\int_0^R f(r)^p w_2(r)dr\right)^{\frac{1}{p}}, \tag{36}$$

holds for all positive functions f on $(0,\infty)$ if and only if

$$D = \sup_{r\in(0,R)} \left(\int_r^R w_1(t)dt\right)^{\frac{1}{q}} \left(\int_0^r w_2(t)^{\frac{1}{1-p}}\,dt\right)^{\frac{p-1}{p}} < \infty.$$

Moreover, the best constant in (36) satisfies the estimate

$$D \le C \le k(p,q)D,$$

where

$$k(p,q) = \left(\frac{p+qp-q}{p}\right)^{\frac{1}{q}} \left(\frac{p+qp-q}{(p-1)q}\right)^{\frac{p-1}{p}}.$$

5 A posteriori error estimates

In this section we first present the *a posteriori error estimators* for the adaptive approximation of problem (4) and then prove their *reliability* and *efficiency*.

The *residual* $\mathcal{R}(V)$ of $V \in \mathbb{V}^\ell_{\mathcal{T}}$ is given by

$$\mathcal{R}(V): W_{-\alpha} \to \mathbb{R}, \qquad \langle \mathcal{R}(V), v\rangle := a(V,v) - \delta_{x_0}(v), \quad \forall\, v \in W_{-\alpha}.$$

Let $U \in \mathbb{V}^\ell_{\mathcal{T}}$ be the solution of the discrete problem (17). Integrating by parts on each $T \in \mathcal{T}$ we have that

$$\langle \mathcal{R}(U), v\rangle = \sum_{T\in\mathcal{T}} \left(\int_T Rv + \int_{\partial T} Jv\right) - \delta_{x_0}(v), \qquad \forall\, v \in W_{-\alpha}, \tag{37}$$

where R denotes the *element residual* given by

$$R_{|_T} := -\nabla\cdot[\mathcal{A}\nabla U] + \mathbf{b}\cdot\nabla U + cU, \qquad \forall\, T \in \mathcal{T},$$

and J the *jump residual* given by

$$J_{|_S} := \frac{1}{2}\left[(\mathcal{A}\nabla U)_{|_{T_1}}\cdot\vec{n}_1 + (\mathcal{A}\nabla U)_{|_{T_2}}\cdot\vec{n}_2\right], \text{ if } S \in \mathcal{E}_\Omega, \quad J_{|_S} = 0, \text{ if } S \in \mathcal{E}_{\partial\Omega}.$$

Here, T_1 and T_2 denote the elements of $\mathcal{T}$ sharing S, and $\vec{n}_1$ and $\vec{n}_2$ are the outward unit normals of T_1 and T_2 on S, respectively.

We define the *a posteriori local error estimator* η_T by

$$\eta_T^2 := \begin{cases} h_T^2 D_T^{2\alpha} \|R\|_{L^2(T)}^2 + h_T D_T^{2\alpha} \|J\|_{L^2(\partial T)}^2 + h_T^{2\alpha+2-n}, & \text{if } x_0 \in T \\ h_T^2 D_T^{2\alpha} \|R\|_{L^2(T)}^2 + h_T D_T^{2\alpha} \|J\|_{L^2(\partial T)}^2, & \text{if } x_0 \notin T \end{cases} \tag{38}$$

and the *global error estimator* η by $\eta := \left(\sum_{T\in\mathcal{T}} \eta_T^2 \right)^{\frac{1}{2}}$.

5.1 Reliability

We first prove the reliability of the global error estimator.

Theorem 5.1 (Global upper bound)**.** *Let $\alpha \in \mathbb{I}$ and let $u \in W_\alpha$ be the solution of problem (4) and let $U \in \mathbb{V}_\mathcal{T}^\ell$ be the solution of the discrete problem (17). Then, there exists a constant $C_\mathcal{U} > 0$ depending on the problem data, the mesh regularity κ and the parameter α such that*

$$\|U - u\|_{H^1_\alpha(\Omega)} \le C_\mathcal{U}\, \eta.$$

The constant $C_\mathcal{U}$ blows up when α approaches an endpoint of $\mathbb{I}$.

The proof follows the usual steps for proving the reliability of residual-type a posteriori error estimators, making use, as in [2], of the continuous inf-sup condition, instead of the usual coercivity. It is strongly based on the weighted estimates and the properties of the quasi-interpolation operator $\mathcal{P}$ stated in the previous section. Recall that $\mathcal{P}$ can be either the Clément or the Scott-Zhang interpolation operator.

Proof. Let $u \in W_\alpha$ be the solution of problem (4) and $U \in \mathbb{V}_\mathcal{T}^\ell$ be the solution of the discrete problem (17). Using the inf-sup condition (16) we have that

$$\frac{1}{C_*}\|U - u\|_{W_\alpha} \le \sup_{v\in W_{-\alpha}} \frac{a(U-u, v)}{\|v\|_{W_{-\alpha}}} = \sup_{v\in W_{-\alpha}} \frac{\langle \mathcal{R}(U), v\rangle}{\|v\|_{W_{-\alpha}}} = \|\mathcal{R}(U)\|_{(W_{-\alpha})'}. \tag{39}$$

Now, let $v \in W_{-\alpha}$ and let $V = \mathcal{P}v$, with $\mathcal{P}$ either the Clément or the Scott-Zhang interpolation operator. Then, by (17), (37) and Hölder inequality it follows that

$$\begin{aligned} |\langle \mathcal{R}(U), v\rangle| &= |\langle \mathcal{R}(U), v - V\rangle| \\ &\le \sum_{T\in\mathcal{T}} \Big(\|R\|_{L^2(T)} \|v - V\|_{L^2(T)} + \|J\|_{L^2(\partial T)} \|v - V\|_{L^2(\partial T)} \Big) + |\delta_{x_0}(v - V)|. \end{aligned}$$

Applying a scaled trace theorem and the interpolation estimates (31) and (32), for the addition in the right hand side of the last inequality, we have that

$$\begin{aligned} &\sum_{T\in\mathcal{T}} \Big(\|R\|_{L^2(T)} \|v - V\|_{L^2(T)} + \|J\|_{L^2(\partial T)} \|v - V\|_{L^2(\partial T)} \Big) \\ &\qquad\lesssim \sum_{T\in\mathcal{T}} \Big(\|R\|_{L^2(T)}\, h_T \|\nabla v\|_{L^2(\omega_T)} + \|J\|_{L^2(\partial T)}\, h_T^{\frac{1}{2}} \|\nabla v\|_{L^2(\omega_T)} \Big) \\ &\qquad\lesssim \sum_{T\in\mathcal{T}} \Big(h_T D_T^\alpha \|R\|_{L^2(T)} + h_T^{\frac{1}{2}} D_T^\alpha \|J\|_{L^2(\partial T)} \Big) \|\nabla v\|_{L^2_{-\alpha}(\omega_T)}, \end{aligned}$$

and using the local bound for the Dirac delta (35), and the weighted interpolation estimates (33) and (34),

$$|\delta_{x_0}(v - V)| \lesssim C_I C_\alpha h_{T_0}^{\alpha + \frac{2-n}{2}} \|\nabla v\|_{L^2_{-\alpha}(\omega_{T_0})},$$

where T_0 is any element containing x_0. Thus, recalling the definition of the error estimators (38),

$$|\langle \mathcal{R}(U), v\rangle| \lesssim C_I C_\alpha \eta \|v\|_{W_{-\alpha}}.$$

Therefore, the last estimation and (39) yield the desired assertion. $\square$

5.2 Efficiency

The proof of the lower bound follows the usual steps using a bubble function to test the residual. We first construct bubble functions and then prove the necessary estimates in Lemma 5.2.

Bubble function for the interior residual estimate. Given $T \in \mathcal{T}$, the goal is to construct a bubble function with its support in T of size $\simeq h_T^n$ and at distance $\gtrsim D_T$ of x_0. To do this, we divide each edge of T into four equal segments and consider the simplices which are determined by one vertex of T and the segments that touch it (see Figure 1). We then let T_* be the one of these simplices that is farthest from x_0, so that

$$h_T \lesssim d_{T_*} := \min_{x \in T_*} \mathrm{d}_{x_0}(x).$$

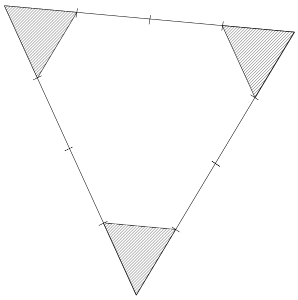

Figure 1: Simplex T and equivalent (shaded) sub-simplices, obtained after dividing the edges into four equal segments. T_* is the one which is farthest from x_0 in order to guarantee that $D_T \lesssim d_{T_*}$.

Since $D_T \simeq d_T \le d_{T_*}$ for $T \in \mathcal{T}^{\mathrm{far}}$, and $D_T \simeq h_T \lesssim d_{T_*}$ for $T \in \mathcal{T}^{\mathrm{near}}$ (cf. Lemma 4.1), we conclude that

$$D_T \lesssim d_{T_*}, \quad \forall T \in \mathcal{T}.$$

Besides, by translating and scaling a fixed bubble function $\hat{\varphi}$ to the sub-element T_* we obtain $\varphi_T \in C_0^\infty(\mathbb{R}^n)$ with

$$\delta_{x_0}(\varphi_T) = \varphi_T(x_0) = 0, \qquad \mathrm{supp}(\varphi_T) \subset T_*, \qquad \|\varphi_T\|_{L^\infty(T)} = 1. \tag{40}$$

Bubble function for the jump residual estimate. Given $S \in \mathcal{E}_\Omega$, we denote T, T' the two elements sharing S. The goal is now to construct a bubble function with its support in ω_S of size $\simeq h_T^n$ and at distance $\gtrsim D_T$ of x_0. We proceed as before, dividing the edges of T and T' into four equal segments. We then consider the simplices determined by the vertices of S and the segments that touch them. This determines n patches of adjacent simplices. We then choose $T_* \subset T$ and $T'_* \subset T'$ such that $T_* \cap T'_* =: S_* \neq \emptyset$ and

$$h_T \lesssim d_{T_*} \qquad \text{and} \quad h_{T'} \lesssim d_{T'_*},$$

the situation for $n = 2$ is depicted in Figure 2.

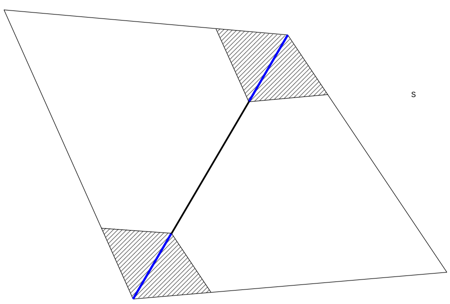

Figure 2: Triangles T, T' sharing a common side S. The patch $T_* \cup T'_*$ is one of the shaded regions, the one farther from x_0, and $S_* = T_* \cap T'_*$. Therefore $D_T \lesssim d_{T_*}$ and $D_{T'} \lesssim d_{T'_*}$.

By construction, we have

$$D_T \lesssim d_{T_*} \quad \text{and} \quad D_{T'} \lesssim d_{T'_*}.$$

In fact, if $T \in \mathcal{T}^{\mathrm{near}}$, $D_T \simeq h_T \lesssim d_{T_*}$, and if $T \in \mathcal{T}^{\mathrm{far}}$, $D_T \simeq d_T \le d_{T_*}$. Analogously, the estimate for T' holds.

By translating and scaling a fixed bubble function $\hat{\varphi}$ to S_* we obtain $\varphi_S \in C_0^\infty(\mathbb{R}^n)$ such that

$$\delta_{x_0}(\varphi_S) = \varphi_S(x_0) = 0, \quad \mathrm{supp}(\varphi_S) \subset T_* \cup T'_* \subset \omega_S, \quad \|\varphi_S\|_{L^\infty(\omega_S)} = 1. \tag{41}$$

The following result summarizes the properties of the just defined bubble functions φ_T and φ_S that we need to prove the efficiency of the local error estimators.

Lemma 5.2. *Let $0 < \alpha < \frac{n}{2}$ and $T \in \mathcal{T}$. If φ_T is the bubble function satisfying (40), then,*

$$\|p\varphi_T\|_{L^2_{-\alpha}(T)} \lesssim D_T^{-\alpha} \|p\|_{L^2(T)}, \tag{42}$$

$$h_T \|\nabla(p\varphi_T)\|_{L^2_{-\alpha}(T)} \lesssim D_T^{-\alpha} \|p\|_{L^2(T)}, \tag{43}$$

for all $p \in \mathcal{P}_{\ell-1}(T)$. On the other hand, if $S \in \mathcal{E}_\Omega$ is a side of T and φ_S is the bubble function satisfying (41), then,

$$h_T^{-\frac{1}{2}} \|p\varphi_S\|_{L^2_{-\alpha}(\omega_S)} \lesssim D_T^{-\alpha} \|p\|_{L^2(S)}, \tag{44}$$

$$h_T^{\frac{1}{2}} \|\nabla(p\varphi_S)\|_{L^2_{-\alpha}(\omega_S)} \lesssim D_T^{-\alpha} \|p\|_{L^2(S)}, \tag{45}$$

for all $p \in \mathcal{P}_{\ell-1}(S)$, where we extend p to ω_S as constant along the direction of one side of each element of $\mathcal{T}$ contained in ω_S.

Proof. $\boxed{1}$ Using that $\|\varphi_T\|_{L^\infty(T)} = 1$ and $\mathrm{supp}(\varphi_T) \subset T_*$, it follows that $\|p\varphi_T\|^2_{L^2_{-\alpha}(T)} = \int_{T_*} p^2 \varphi_T^2 \, \mathrm{d}_{x_0}^{-2\alpha} \leq d_{T_*}^{-2\alpha} \|p\|^2_{L^2(T)}$. Taking into account that $D_T \lesssim d_{T_*}$, (42) holds.

$\boxed{2}$ The usual scaling arguments yield

$$\|\nabla(p\varphi_T)\|_{L^2(T)} \simeq h_T^{-1} \|p\|_{L^2(T)}, \quad \forall\, p \in \mathcal{P}_{\ell-1}(T),$$

and thus

$$\|\nabla(p\varphi_T)\|^2_{L^2_{-\alpha}(T)} = \int_{T_*} |\nabla(p\varphi_T)|^2 \, \mathrm{d}_{x_0}^{-2\alpha} \lesssim d_{T_*}^{-2\alpha} \|\nabla(p\varphi_T)\|^2_{L^2(T)} \lesssim d_{T_*}^{-2\alpha} h_T^{-2} \|p\|^2_{L^2(T)}.$$

In consequence, (43) follows from $D_T \lesssim d_{T_*}$.

$\boxed{3}$ Let $T \in \mathcal{T}$ be such that $S \subset T \subset \omega_S$. Since $\|\varphi_S\|_{L^\infty(\omega_S)} = 1$ and $\mathrm{supp}(\varphi_S) \subset T_* \cup T'_*$, we have

$$\|p\varphi_S\|^2_{L^2_{-\alpha}(T)} = \int_{T_*} p^2 \varphi_S^2 \, \mathrm{d}_{x_0}^{-2\alpha} \leq d_{T_*}^{-2\alpha} \int_{T_*} p^2 \lesssim d_{T_*}^{-2\alpha} h_T \int_{S_*} p^2 \leq d_{T_*}^{-2\alpha} h_T \|p\|^2_{L^2(S)},$$

and therefore, (44) holds, using that $D_T \lesssim d_{T_*}$.

$\boxed{4}$ Let $T \in \mathcal{T}$ be such that $S \subset T \subset \omega_S$. The usual scaling arguments yield

$$\|\nabla(p\varphi_S)\|_{L^2(T)} \simeq h_T^{-1} \|p\|_{L^2(T \cap \mathrm{supp}(\varphi_S))}, \quad \forall\, p \in \mathcal{P}_{\ell-1}(T).$$

Let us denote by T_* the element which is contained in T (cf. Figure 2). Hence

$$\begin{aligned}\|\nabla(p\varphi_S)\|^2_{L^2_{-\alpha}(T)} &= \int_{T_*} |\nabla(p\varphi_S)|^2 \, \mathrm{d}_{x_0}^{-2\alpha} \lesssim d_{T_*}^{-2\alpha} \|\nabla(p\varphi_S)\|^2_{L^2(T)} \\ &\simeq d_{T_*}^{-2\alpha} h_T^{-2} \|p\|^2_{L^2(T_*)} \lesssim d_{T_*}^{-2\alpha} h_T^{-1} \|p\|^2_{L^2(S_*)} \leq d_{T_{S_*}}^{-2\alpha} h_T^{-1} \|p\|^2_{L^2(S)}.\end{aligned}$$

Finally, (45) follows due to $D_T \lesssim d_{T_*}$. $\square$

As usually happens for residual based error estimators, the lower bound is local, and holds up to some oscillation terms. In this context, we define the *local oscillation* osc_T by

$$\mathrm{osc}_T := \begin{cases} \left(h_T^2 D_T^{2\alpha} \left\|R - \overline{R}\right\|^2_{L^2(\omega_T)} + h_T D_T^{2\alpha} \left\|J - \overline{J}\right\|^2_{L^2(\mathcal{E}_\Omega \cap (\omega_T)^0)} \right)^{\frac{1}{2}}, & \textit{if } x_0 \in T, \\ \left(h_T^2 D_T^{2\alpha} \left\|R - \overline{R}\right\|^2_{L^2(\omega_T)} + h_T D_T^{2\alpha} \left\|J - \overline{J}\right\|^2_{L^2(\partial T)} \right)^{\frac{1}{2}}, & \textit{if } x_0 \notin T, \end{cases}$$

where $\overline{R}_{|_{T'}}$ denotes the L^2 projection of R on $\mathcal{P}_{\ell-1}(T')$, for all $T' \in \mathcal{T}$, and for each side S, $\overline{J}_{|_S}$ denotes the L^2 projection of J on $\mathcal{P}_{\ell-1}(S)$. Notice that if $x_0 \in T$ the jump oscillations are considered over all $S \in \mathcal{E}_\Omega$ that touch T, including those contained in ∂T and those not contained in ∂T.

The next result is usually called *local efficiency of the error estimator*, based on the fact that whenever a local estimator is large, so is the corresponding local error, provided the local oscillation is relatively small. Its proof follows the usual techniques taking into account the bounds from the last lemma and the boundedness of the bilinear form, yielding

$$|\langle \mathcal{R}(U), v\rangle| = |a(U,v) - \delta_{x_0}(v)| = |a(U,v) - a(u,v)| \leq C_a \|U-u\|_{H^1_\alpha(\omega)} \|v\|_{H^1_{-\alpha}(\omega)},$$

for all $v \in W_{-\alpha}$ with $\operatorname{supp}(v) \subset \omega$, for any $\omega \subset \overline{\Omega}$. Here, the constant $C_a > 0$ depends on the coefficients of the bilinear form a.

Theorem 5.3 (Local lower bound)**.** *Let $\alpha \in \mathbb{I}$, let $u \in W_\alpha$ be the solution of problem (4) and let $U \in \mathbb{V}^\ell_{\mathcal{T}}$ be the solution of the discrete problem (17). There exists a constant $C_{\mathcal{L}} > 0$ depending on the problem data, the mesh regularity κ and the parameter α such that*

$$C_{\mathcal{L}} \eta_T \leq \|U-u\|_{H^1_\alpha(\omega_T)} + \operatorname{osc}_T,$$

for all $T \in \mathcal{T}$. The constant $C_{\mathcal{L}}$ goes to zero if α approaches $\frac{n}{2}$.

Proof. $\boxed{1}$ Let $T \in \mathcal{T}$. We analize first the residual R. Since

$$\|R\|_{L^2(T)} \leq \left\|\overline{R}\right\|_{L^2(T)} + \left\|R - \overline{R}\right\|_{L^2(T)}, \tag{46}$$

it is sufficient to estimate $\left\|\overline{R}\right\|_{L^2(T)}$.

Let φ_T be the bubble function satisfying (40). The usual scaling arguments yield

$$\|\overline{R}\|^2_{L^2(T)} \simeq \left\|\overline{R}\varphi_T^{\frac{1}{2}}\right\|^2_{L^2(T)} = \int_T \overline{R}^2 \varphi_T = \int_T \overline{R} v = \int_T Rv + \int_T (\overline{R} - R) v, \tag{47}$$

where $v := \overline{R}\varphi_T$. Since $\operatorname{supp}(v) \subset T$ and $\delta_{x_0}(v) = 0$, the first integral in the right-hand side of (47), using (42) and (43) satisfies

$$\int_T Rv = \langle \mathcal{R}(U), v\rangle \leq C_a \|U-u\|_{H^1_\alpha(T)} \|v\|_{H^1_{-\alpha}(T)} \lesssim C_a h_T^{-1} \|U-u\|_{H^1_\alpha(T)} D_T^{-\alpha} \left\|\overline{R}\right\|_{L^2(T)},$$

while the second one satisfies

$$\int_T (\overline{R} - R) v \leq \left\|\overline{R} - R\right\|_{L^2(T)} \|v\|_{L^2(T)} \leq \left\|\overline{R} - R\right\|_{L^2(T)} \left\|\overline{R}\right\|_{L^2(T)}.$$

Using the two last inequalities in (47) we have that

$$h_T D_T^\alpha \left\|\overline{R}\right\|_{L^2(T)} \lesssim C_a \|U-u\|_{H^1_\alpha(T)} + h_T D_T^\alpha \left\|\overline{R} - R\right\|_{L^2(T)}. \tag{48}$$

Finally, from (46) and (48) it follows that

$$h_T D_T^\alpha \|R\|_{L^2(T)} \lesssim C_a \|U-u\|_{H^1_\alpha(T)} + h_T D_T^\alpha \left\|\overline{R} - R\right\|_{L^2(T)}. \tag{49}$$

$\boxed{2}$ Secondly, we estimate the jump residual J. Let S be a side of T. As before, it is sufficient to bound the projection $\overline{J}$ of J, since

$$\|J\|_{L^2(S)} \leq \left\|\overline{J}\right\|_{L^2(S)} + \left\|J - \overline{J}\right\|_{L^2(S)}. \tag{50}$$

Let φ_S be the bubble function from (41). Then, usual scaling arguments lead to

$$\|\overline{J}\|^2_{L^2(S)} \lesssim \left\|\overline{J}\varphi_S^{\frac{1}{2}}\right\|^2_{L^2(S)} = \int_S \overline{J}^2 \varphi_S = \int_S \overline{J} v = \int_S Jv + \int_S (\overline{J} - J) v, \tag{51}$$

with $v := \overline{J}\varphi_S$. Extending $\overline{J}$ to ω_S as constant along the direction of one side of each element of $\mathcal{T}$ contained in ω_S, using that $\delta_{x_0}(v) = 0$ and $\operatorname{supp}(v) \subset \omega_S$, the first integral in the right-hand side of (51) can be bounded as follows:

$$\begin{aligned}2\int_S Jv &= \langle \mathcal{R}(U), v\rangle - \int_{\omega_S} Rv \\ &\leq C_a \|U-u\|_{H^1_\alpha(\omega_S)} \|v\|_{H^1_{-\alpha}(\omega_S)} + \|R\|_{L^2(\omega_S)} \|v\|_{L^2(\omega_S)} \\ &\lesssim h_T^{-\frac{1}{2}} C_a \|U-u\|_{H^1_\alpha(\omega_S)} D_T^{-\alpha} \left\|\overline{J}\right\|_{L^2(S)} + h_T^{\frac{1}{2}} \|R\|_{L^2(\omega_S)} \left\|\overline{J}\right\|_{L^2(S)},\end{aligned}$$

where in the last inequality we have used (44) and (45). The second integral in the right-hand side of (51), satisfies

$$\int_S (\overline{J} - J)v \leq \left\|\overline{J} - J\right\|_{L^2(S)} \|v\|_{L^2(S)} \lesssim \left\|\overline{J} - J\right\|_{L^2(S)} \left\|\overline{J}\right\|_{L^2(S)}.$$

The last two estimates and (51) yield

$$h_T^{\frac{1}{2}} D_T^\alpha \left\|\overline{J}\right\|_{L^2(S)} \lesssim C_a \|U-u\|_{H^1_\alpha(\omega_S)} + h_T D_T^\alpha \|R\|_{L^2(\omega_S)} + h_T^{\frac{1}{2}} D_T^\alpha \left\|\overline{J} - J\right\|_{L^2(S)}. \tag{52}$$

Thus, from (50) and (52) we have that

$$h_T^{\frac{1}{2}} D_T^\alpha \|J\|_{L^2(S)} \lesssim C_a \|U-u\|_{H^1_\alpha(\omega_S)} + h_T D_T^\alpha \|R\|_{L^2(\omega_S)} + h_T^{\frac{1}{2}} D_T^\alpha \left\|\overline{J} - J\right\|_{L^2(S)}.$$

Adding the last inequality over all the sides $S \subset \partial T$ and using (49) we obtain

$$h_T^{\frac{1}{2}} D_T^\alpha \|J\|_{L^2(\partial T)} \lesssim C_a \|U-u\|_{H^1_\alpha(\omega_T)} + \operatorname{osc}_T. \tag{53}$$

$\boxed{3}$ Recall that if $x_0 \in T$ the indicator η_T contains also a term $h_T^{\alpha+\frac{2-n}{2}}$, we now prove that

$$h_T^{\alpha+\frac{2-n}{2}} \lesssim \left[(\frac{n}{2}-\alpha)^{-\frac{1}{2}} C_a \|U-u\|_{H^1_\alpha(\omega_T)} + \operatorname{osc}_T\right].$$

Let $\phi \in C^\infty(\mathbb{R}^n)$ with $\|\phi\|_{L^\infty} = \phi(0) = 1$ and $\operatorname{supp}(\phi) \subset B(0,1)$. Let $C = c_{\kappa,\Omega}$ from Lemma 4.2 so that $B(x_0, Ch_T) \subset \omega_T$, and if $\varphi(x) := \phi\left(\frac{x-x_0}{Ch_T}\right)$ then $\delta_{x_0}(\varphi) = \varphi(x_0) = 1$, $\|\varphi\|_{L^\infty} = 1$, $\|\nabla\varphi\|_{L^\infty} \lesssim \frac{1}{h_T}$ and $\operatorname{supp}(\varphi) \subset B(x_0, Ch_T) \subset \omega_T$. Thus, we also have that $\|\varphi\|_{L^2(\omega_T)} \lesssim h_T^{\frac{n}{2}}$, $\|\nabla\varphi\|_{L^2(\omega_T)} \lesssim h_T^{\frac{n-2}{2}}$, and using a scaled trace theorem, $\|\varphi\|_{L^2(\partial T)} \lesssim h_T^{\frac{n-1}{2}}$. On the other hand, applying Lemma 5.4 stated below we have that $\|\varphi\|_{L^2_{-\alpha}(\omega_T)} \lesssim (\frac{n}{2}-\alpha)^{-\frac{1}{2}} h_T^{\frac{n}{2}-\alpha}$ and $\|\nabla\varphi\|_{L^2_{-\alpha}(\omega_T)} \lesssim (\frac{n}{2}-\alpha)^{-\frac{1}{2}} h_T^{\frac{n-2}{2}-\alpha}$. Therefore,

$$\begin{aligned}1 &= \delta_{x_0}(\varphi) = a(u,\varphi) = a(u-U,\varphi) + a(U,\varphi) \\ &\leq C_a \|U-u\|_{H^1_\alpha(\omega_T)} \|\varphi\|_{H^1_{-\alpha}(\omega_T)} + \sum_{T' \subset \omega_T} \left(\int_{T'} R\varphi + \int_{\partial T'} J\varphi\right) \\ &\leq C_a \|U-u\|_{H^1_\alpha(\omega_T)} \|\varphi\|_{H^1_{-\alpha}(\omega_T)} + \sum_{T' \subset \omega_T} \|R\|_{L^2(T')} \|\varphi\|_{L^2(T')} + 2 \sum_{S \subset (\omega_T)^0} \|J\|_{L^2(S)} \|\varphi\|_{L^2(S)} \\ &\lesssim \left((\frac{n}{2}-\alpha)^{-\frac{1}{2}} C_a \|U-u\|_{H^1_\alpha(\omega_T)} + \sum_{T' \subset \omega_T} h_{T'} D_{T'}^\alpha \|R\|_{L^2(T')} + \sum_{S \subset (\omega_T)^0} h_{T'}^{\frac{1}{2}} D_{T'}^\alpha \|J\|_{L^2(S)}\right) h_T^{-\alpha+\frac{n-2}{2}}.\end{aligned}$$

The last inequality with the estimates obtained in steps $\boxed{1}$ and $\boxed{2}$ complete the proof. $\square$

Lemma 5.4. *If $0 < \alpha < \frac{n}{2}$ and $T \in \mathcal{T}$, then*

$$\left\|\mathrm{d}_{x_0}^{-\alpha}\right\|_{L^2(T)} \lesssim \frac{1}{\sqrt{\frac{n}{2}-\alpha}} h_T^{\frac{n}{2}-\alpha}.$$

Proof. Let $T \in \mathcal{T}$ and let $\tilde{x}_0 \in T$ such that $\mathrm{d}_{x_0}(\tilde{x}_0) = |\tilde{x}_0 - x_0| = \mathrm{dist}(x_0, T)$. Then, if we define $\mathrm{d}_{\tilde{x}_0}(x) := |x - \tilde{x}_0|$ we have that

$$\mathrm{d}_{\tilde{x}_0}(x) \leq \mathrm{d}_{x_0}(x), \quad \forall\, x \in T,$$

and thus, if $0 < \alpha < \frac{n}{2}$,

$$\left\|\mathrm{d}_{x_0}^{-\alpha}\right\|_{L^2(T)} \leq \left\|\mathrm{d}_{\tilde{x}_0}^{-\alpha}\right\|_{L^2(T)} \leq \left\|\mathrm{d}_{\tilde{x}_0}^{-\alpha}\right\|_{L^2(B(\tilde{x}_0,\kappa h_T))} \lesssim \frac{1}{\sqrt{\frac{n}{2}-\alpha}} h_T^{\frac{n}{2}-\alpha}.$$

□

As an immediate consequence of Theorem 5.3, adding over all elements in the mesh we obtain the efficiency of the global error estimator.

Theorem 5.5 (Global lower bound)**.** *Let $\alpha \in \mathbb{I}$, let $u \in W_\alpha$ be the solution of problem (4) and let $U \in \mathbb{V}_{\mathcal{T}}^{\ell}$ be the solution of the discrete problem (17). There exists a constant $C_L > 0$ depending on the problem data, the mesh regularity κ and the parameter α such that*

$$C_L \eta \leq \|U - u\|_{H^1_\alpha(\Omega)} + \mathrm{osc},$$

where osc *is the* global oscillation *defined by* $\mathrm{osc} := \left(\sum_{T\in\mathcal{T}} \mathrm{osc}_T^2\right)^{\frac{1}{2}}$*, and the constant C_L goes to zero if α approaches $\frac{n}{2}$.*

Remark 5.6 (Convergence of adaptive algorithms)**.** The general convergence theory from [17] states that if the discretization of a linear problem is stable, the a posteriori error estimators constitute an upper bound for the error and if there holds a discrete local lower bound, up to oscillation terms, then any adaptive algorithm marking at least the element with the largest indicator will converge. Our indicators, in the framework of the weighted spaces considered here, fulfill all those assumptions, yielding convergence. For the discrete lower bound it is enough to observe that discrete bubble functions φ_T and φ_S can be constructed on sufficiently refined meshes, so that they satisfy (40), (41) and thus also Lemma 5.2.

6 Numerical experiments

In this section we report some numerical experiments that document the behavior of the adaptive algorithm based on our a posteriori estimators for the error in W_α norm. We implemented a loop of the usual form

$$\textsc{Solve} \longrightarrow \textsc{Estimate} \longrightarrow \textsc{Mark} \longrightarrow \textsc{Refine}.$$

The step Solve consisted in solving the discrete system for the current mesh, the step Estimate consisted in computing the a posteriori error estimators η_T for a given value of α. In the step Mark we selected in $\mathcal{M}$ for refinement those elements $T \in \mathcal{T}$ with largest estimators η_T until $\sum_{T\in\mathcal{M}} \eta_T^2 \geq 0.5 \sum_{T\in\mathcal{T}} \eta_T^2$, i.e., we used the *Dörfler strategy* with parameter 0.5. The step Refine consisted in performing two bisections to each marked element, and refining some extra elements in order to keep conformity of the meshes, using the *newest-vertex bisection*. We used a custom implementation in MATLAB.

We present two examples on two-dimensional domains, using piecewise linear finite elements. The first one considering a known solution on an L-shaped domain, and the second one based on the computation of an unknown solution on a rectangle, with variable coefficients, simulating a wiggling flow on a canal.

Example 1. We consider the boundary value problem $-\Delta u = \delta_{(0.5,0.5)}$ in the L-shaped domain $\Omega = (-1,1)^2 \setminus [0,1) \times (-1,0] \subset \mathbb{R}^2$ with exact solution $u(x) = -\frac{1}{2\pi} \log|x - (0.5, 0.5)| + |x|^{2/3} \sin(2\theta/3)$, ($\theta$ the angle measured from 0 to $3\pi/2$ in Ω), and Dirichlet boundary conditions.

The goal of this example is to test the behavior of the adaptive method guided by the a posteriori estimators η_T for different values of α, in a problem with two singularities. One produced by the Dirac delta on the right-hand side and another one produced by the reentrant corner. Our theory predicts that $\eta := \left(\sum_{T\in\mathcal{T}} \eta_T^2\right)^{1/2}$ is equivalent to the error in W_α norm provided $0 < \alpha < 1$.

In Figure 3 we show the decay of the W_α and the $L^2(\Omega)$ norm of the error $u-U$, versus the number of Degrees of Freedom (DOFs) in logarithmic scales, for $\alpha = 0.1, 0.3, 0.5, 0.7, 0.9$. We observe the optimal decay $(\#\mathcal{T})^{-1/2}$ and $(\#\mathcal{T})^{-1}$, respectively. This is the same decay proved by D'Angelo for properly graded meshes [6], making use of the cardinality results from [9]. As is usual with adaptive methods, the optimal cardinality is obtained automatically, without any fine tuning or additional requirement on the meshes.

We also plot the effectivity index $\|u-U\|_{W_\alpha}/\eta$ and observe that it remains between 0.12 and 0.35 for all the considered values of α, showing the robustness of the estimator with respect to α, with no degeneracy as α approaches the endpoints of $\mathbb{I}$. This is better than expected according to our theory.

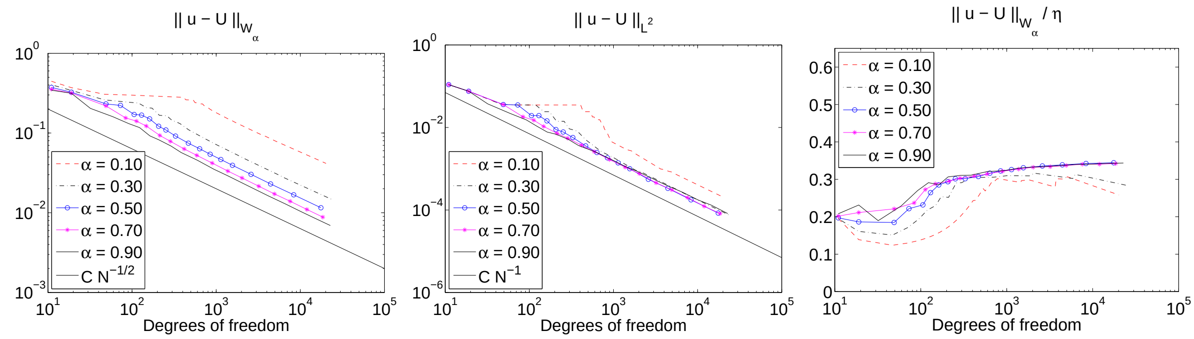

Figure 3: Exact errors and effectivity indices for Example 1. We plot the W_α (left) and the $L^2(\Omega)$ (middle) norm of the error $u-U$ versus the number of Degrees of Freedom (DOFs) in logarithmic scales, for different values of α. We observe the optimal decay $(\#\mathcal{T})^{-1/2}$ and $(\#\mathcal{T})^{-1}$, respectively. We also plot the effectivity index $\|u-U\|_{W_\alpha}/\eta$ and observe that it remains between 0.12 and 0.35 for all the considered values of α, showing the robustness of the estimator with respect to α.

In Figure 4 we show the decay of the W_α and the $L^2(\Omega)$ norm of the error $u-U$, for values of α very close to zero. We show the behavior for $\alpha = 0.05, 0.1, 0.15, 0.2$ and observe the optimal decays for the cases $\alpha \geq 0.1$. The algorithm stopped after 53 iterations in the case $\alpha = 0.05$, with a mesh of 2544 elements and 1286 degrees of freedom (DOFs). The refinement is concentrated solely around the support of the Dirac delta, leading to very small elements, with diameter of order 2^{-53}. The resulting system matrix was singular to working precision. We also show the effectivity indices for these values of α and observe that they do not degenerate as α approaches zero.

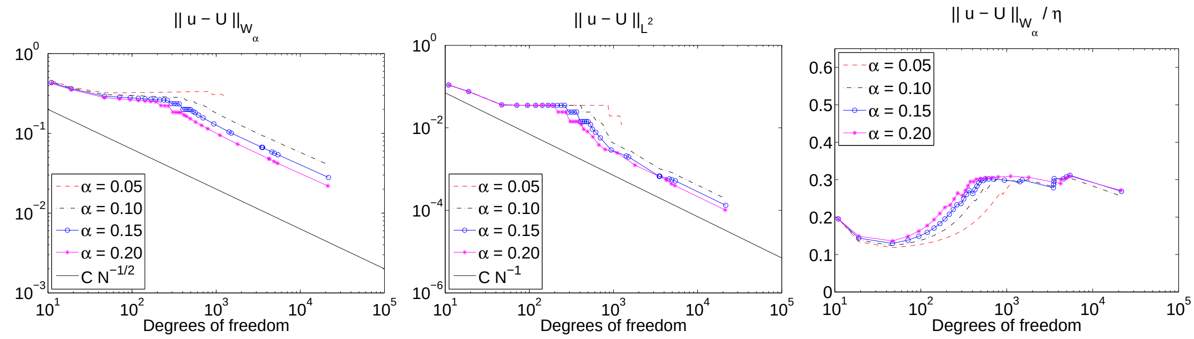

Figure 4: Exact errors and effectivity indices for Example 1 and α very small. We plot the W_α (left) and the $L^2(\Omega)$ (middle) norm of the error $u-U$ versus the number of Degrees of Freedom (DOFs) in logarithmic scales, for different values of α. We observe the optimal decay for all the considered values, except for the smallest value $\alpha = 0.05$. In this extreme situation the algorithm refines purely around $(0.5, 0.5)$ and the elements become excessively small, leading to a nearly singular system matrix (to the working precision) not allowing computation beyond a mesh with 2544 elements and 1286 DOFs, obtained after 53 iterations. The effectivity index $\|u-U\|_{W_\alpha}/\eta$, plotted on the right, remains bounded between 0.11 and 0.32.

The meshes after 4, 8 and 12 iterations for $\alpha = 0.25, 0.5, 0.75$ are plotted in Figure 5. The number of elements of the corresponding meshes is indicated in each picture, and the stronger grading obtained for smaller values of α is not so apparent for these values of α, although the case $\alpha = 0.25$ is much different than the other two cases. It is worth observing that the corner singularity is not noticed for $\alpha = 0.25$ after 8 iterations of the adaptive algorithm, and it is immediately noticed for α bigger (see also Figure 6).

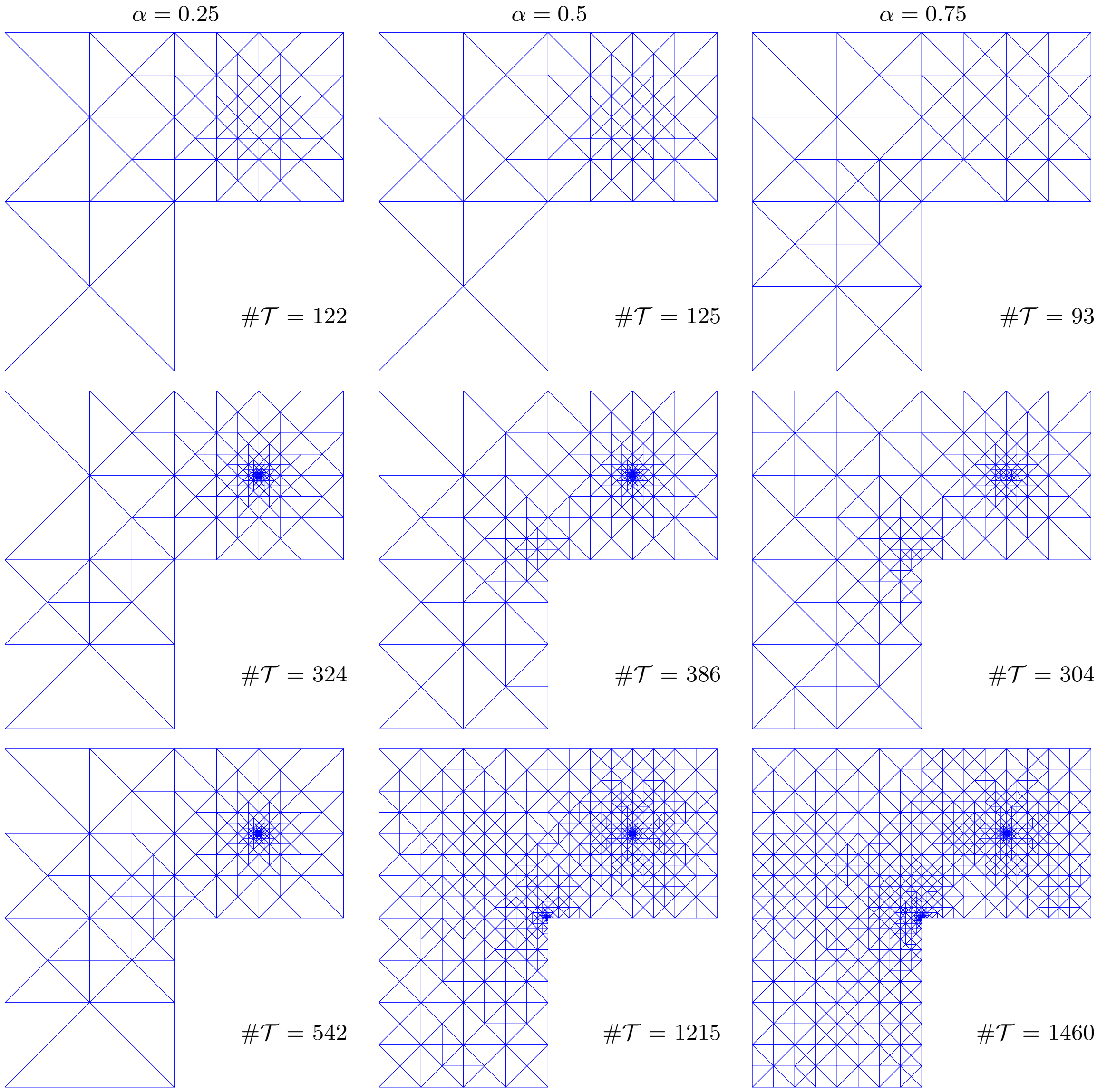

Figure 5: Meshes for Example 1. We show the meshes after 4 (top), 8 (middle) and 12 (bottom) iterations for $\alpha = 0.25$ (left), $\alpha = 0.5$ (middle) and $\alpha = 0.75$ (right). The number of elements of the corresponding meshes is indicated in each picture, and the stronger grading obtained for smaller values of α is not so clearly visible. It is worth observing that the corner singularity is not noticed at all for $\alpha = 0.25$ after 8 iterations of the adapive algorithm, and barely after 12 iterations, but it is immediately noticed for α big. In the latter case the refinement is more spread throughout the domain, due to the smaller relative importance of the singularity introduced by δ_{x_0}.

We also plot meshes with a similar number of elements for values of $\alpha = 0.1, 0.3, 0.5$ in Figure 7. The fact that the singularity introduced by the Dirac delta is less severe when the error is measured in W_α for bigger α is noticeable in this picture. The refinement is thus more spread in this case.

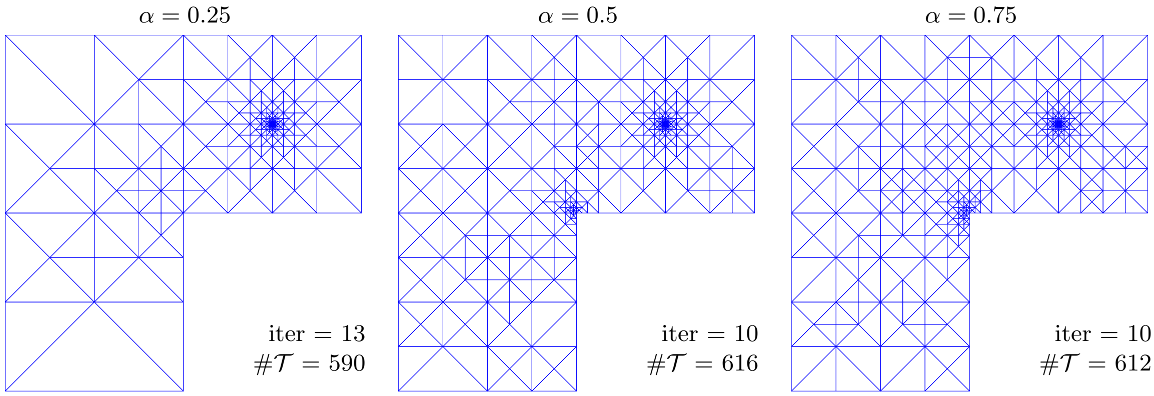

Figure 6: Meshes for Example 1 with similar number of elements. We show meshes for different values of α and similar number of elements. There is no significant difference for the values of $\alpha = 0.5, 0.75$.

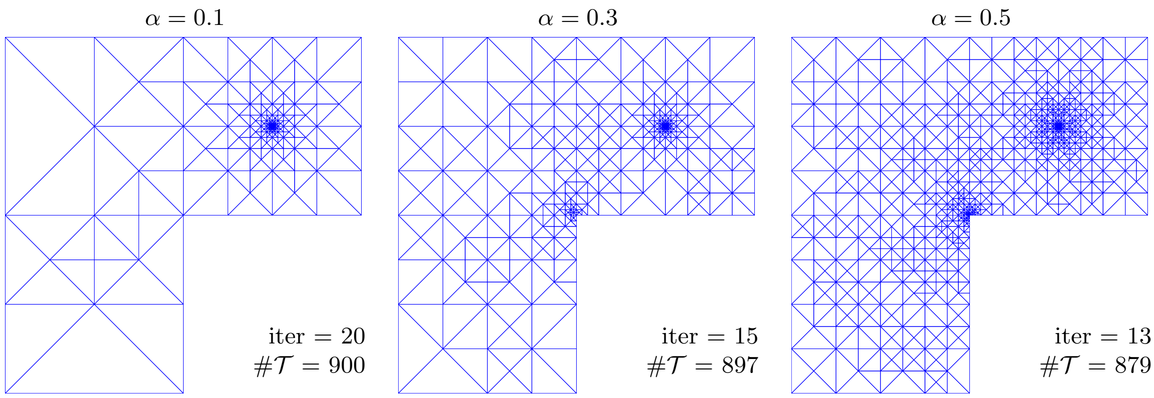

Figure 7: Meshes for Example 1 with similar number of elements. We show meshes for different values of α close to zero and similar number of elements. We can observe that for smaller values of α the meshes are more strongly graded at $(0.5, 0.5)$ where the Dirac delta is supported. For α big the algorithm notices early the presence of the corner singularity.

Example 2. In this example we let $\Omega = (0,3)\times(0,1)$ and consider the problem

$$\begin{aligned} -0.02\,\Delta u + \begin{bmatrix} 2 \\ \sin(5x_1) \end{bmatrix} \cdot \nabla u + 0.1u &= \delta_{(0.2,0.4)} && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \cap \{x_1 < 3\}, \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \partial\Omega \cap \{x_1 = 3\}, \end{aligned}$$

which is a diffusion-advection-reaction equation, typical from pollutant transport and degradation in aquatic media.

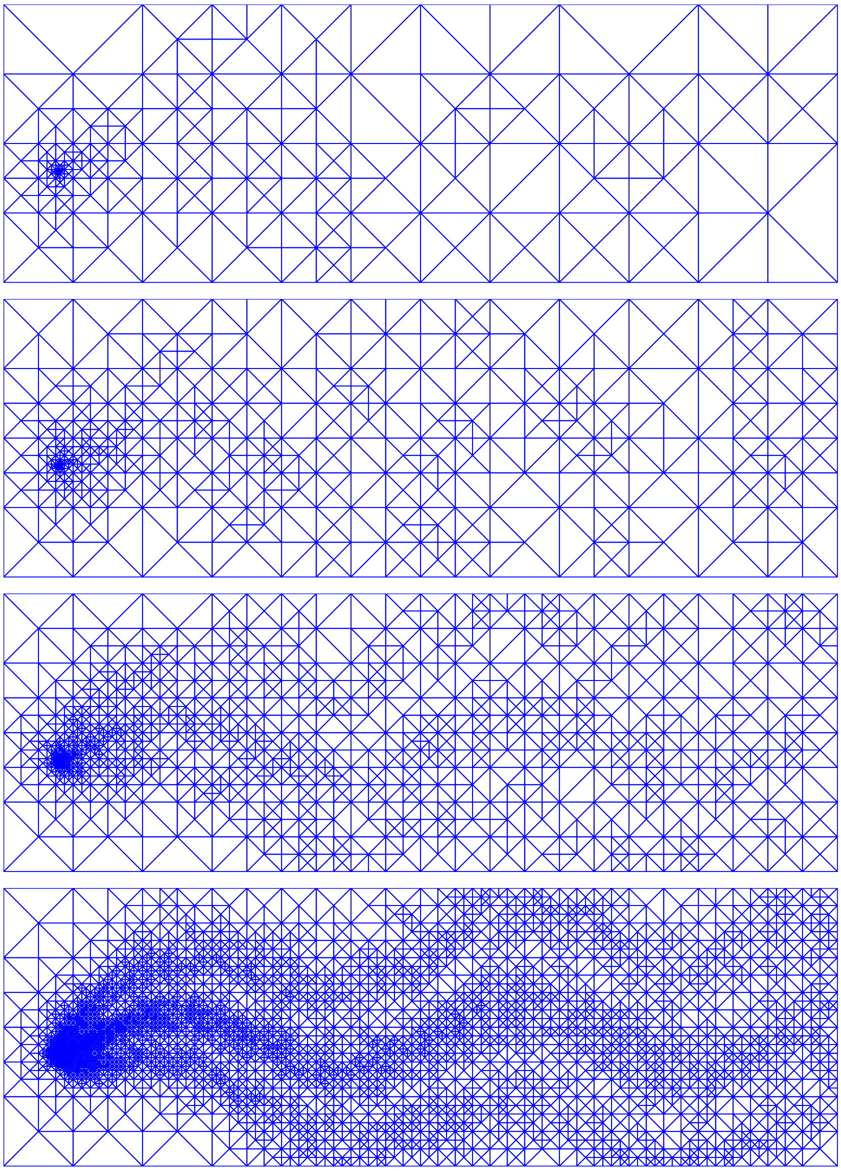

Figure 8: Meshes for Example 2. We show the meshes obtained by the adaptive loop after 10, 13, 16 and 19 iterations, with 398, 918, 2409 and 10608 elements, respectively.

A sequence of meshes is presented in Figure 8. The solution in the final mesh, with 22256 elements and 11212 DOFs can be observed in Figure 9. The computation was done with the same adaptive algorithm of the previous example, with $\alpha = 0.5$.

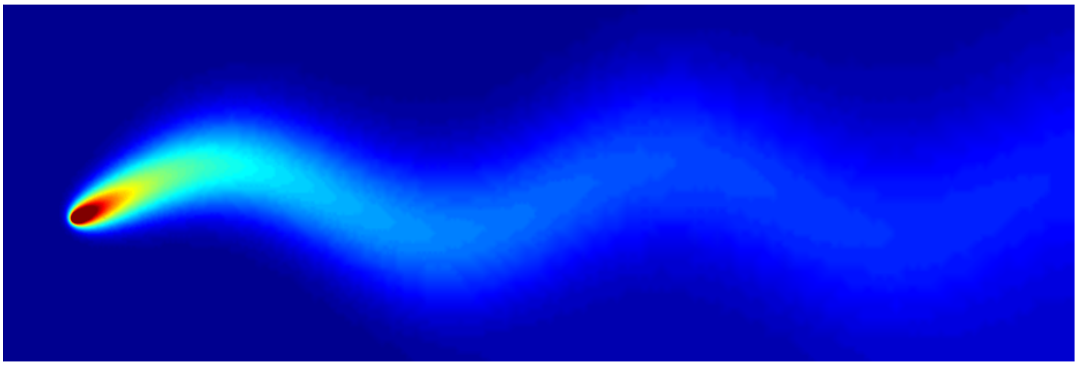

Figure 9: Final solution for Example 2, obtained by the adaptive loop after 20 iterations, on a mesh with 22256 elements and 11212 DOFs. The error estimator for this mesh is 0.024, which is a 2.2% of the estimator for the initial coarsest mesh.

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